Optimal City Structure *

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First Version: July 2015

PRELIMINARY AND INCOMPLETE

Abstract

In this paper, we develop a quantitative general equilibrium model of a city that incorporates the many economic interactions that occur over the space of the city, including commuting, trade, and productive interactions. We show that despite the many spatial linkages, in the absence of externalities the competitive equilibrium is efficient; conversely, in the presence of spillovers, there exists opportunities for a city planner to increase the welfare of the city inhabitants by restricting the use of land ("zoning"). We provide sufficient conditions for the optimal zoning policy that depend solely on observables and several key model parameters. Finally, we illustrate the flexibility of the model by applying it to study the observed zoning policy of the city of Chicago. Preliminary results suggest that the welfare of Chicago residents would increase if more area was allocated to residential usage in the central business district and more area was allocated to businesses in the outlying neighborhoods.

*We thank Dave Donaldson, Davin Chor, Esteban Rossi-Hansberg, Matthew Turner, Yang Yao and Xiaobo Zhang. All errors are our own.

1 Introduction

As of 2014, 54% of people worldwide live in cities. This is an increase from 34% in 1960, and urban population is expected to increase by more than 1% per year in the upcoming decades. This unprecedented concentration is indicative of the large agglomeration economies that take place in shorter distances and lead firms and individuals to cluster in cities. While local governments have a large array of potential policy tools at their disposal (e.g. zoning policies, subsidies, infrastructure projects, etc.), little is known about how a city can best take advantage of these agglomeration economies in order to improve the welfare of its citizens.

In this paper, we develop a quantitative general equilibrium model of a city that incorporates the spatial linkages present in modern cities and the many ways individuals interact with each other across space. Our model allows us to examine the general equilibrium effects of different policies on agents welfare and to quantify their impact. As in trade and economic geography models, we model the movement of goods from firms to consumers to be subject to bilateral trade costs. As in urban models, we also assume there is a cost for individuals to move around the city. This cost of moving throughout the city has important ramifications for the many economic choices that individuals living in a city make, including where to live, where to work, and with whom to interact (which affects an agent's productivity). These decisions determine the distribution of wages, productivity, output, and land prices throughout the city and ultimately the welfare of those living in the city.

We first prove that in the absence of productivity externalities, the competitive equilibrium is efficient, so that there is no need for any intervention on the part of the city planner. While this result is not particularly surprising given the standard first-welfare theorem, it is reassuring that efficiency is robust to the many spatial linkages present in the model. Conversely, when there does exist productivity externalities, a city planner can improve the welfare of inhabitants by imposing restrictions on land use by zoning.

We then present a parameterized "quantitative" version of the model. Like Ahlfeldt, Redding, Sturm, and Wolf (2012), we assume that individuals have idiosyncratic Frechet distributed preferences which give a convenient "gravity" form for commuting flows. As in Anderson (1979) and Allen and Arkolakis (2014) we assume that each location produces a differentiated product. We further assume that firms produce the good by combining land and labor in a Cobb-Douglas fashion, and as in Helpman (1998) and Redding (2015), individuals have Cobb-Douglas preferences over housing and the traded good.

We allow agents to optimally allocate their time across a number of activities. Individuals can spend time working and commuting (to earn income to purchase goods), in leisure (from which they get utility directly), or in interacting with others in the city (from which they generate ideas and increase their own productivity). It is this interaction, which has been emphasized previously by Charlot and Duranton (2004), Glaeser (1999), and Davis and Dingel (2012) (amongst others) but has yet to be incorporated into a quantitative framework, that generates productivity externalities and opens up the possibility for interventions by the city planner to be welfare improving.

The model incorporates the main links that trade, geography and urban economists identify: flows of trade, flows of commuting, and spatial spillovers of knowledge. Despite its richness, the structure remains suprisingly tractable. To characterize the solution of the problem we consider the concept of a zoning equilibrium where the use of land for residential and commercial purposes is specified by a city planner. In this equilibrium good, labor and rental markets in the model clear and the model aggregates to a system of equations. Exploiting results from Allen, Arkolakis, and Li (2015) we can provide a general characterization of existence and uniqueness in the model.[1] In addition, we show that if trade costs are assumed to be zero, the model extends the standard equilibrium equations arising from spatial models to the case of commuting. In particular, the number of workers in a location and effective wages are not affected only by location fundamentals, such as productivity and land, but also from a multilateral commuting accessibility term that determines the accessibility of a production location from high amenity, easy to access residential locations, were commuters can live. In addition, under certain geographies, such as the circle, and under symmetric fundamentals the model allows for an explicit solution of the population density.

Given this structure, we derive the elasticity of the welfare of the city inhabitants to a change in the zoning in any location within the city, thereby providing a necessary condition for the optimal zoning of a city. Conveniently, the elasticity depend only on the observed economic activity of the city (e.g. commuting flows) and the structural parameters governing the strength of the general equilibrium forces in the model, and can be simultaneously determined for all locations by a single matrix inversion. Even if the necessary condition is not satisfied, its sign indicates how a location should be re-zoned and its magnitude indicates the gains in welfare from doing so.

Finally, we illustrate how the model can be brought to real world using data from the city of Chicago. Combining data on commuting flows, bilateral travel times, zoning, and the location, height and size of 820,944 buildings in the city, we show how to identify the unobserved distribution of productivities and amenities throughout the city. By the theoretical results above, we estimate that there are substantial gains from re-zoning in Chicago: in

[1] Monte, Redding, and Rossi-Hansberg (2015) also consider a model of commuting with labor mobility and trade, although they do not allow for the possibility of productivity spillovers or solve for agent's equilibrium time use. They too provide conditions for existence and uniqueness of the equilibrium with fixed land use, but their conditions and their approach to proving uniqueness are somewhat different.

particular, the model implies that welfare could be improved if the central business district re-zoned to more residential units and the residential areas in neighborhoods outside the center of Chicago re-zoned to allow more commercial units.

Our approach merges the standard general equilibrium analysis of welfare used in trade and geography models (see, for example, Arkolakis, Costinot, and Rodríguez-Clare (2012) and Allen and Arkolakis (2014)) to understand the impact of local policies on welfare in an urban setting. There are two distinct advantages to our approach: First, because we can provide a sharp characterization of existence and uniqueness we can analyze the impact of local policies by providing explicit ways to compute changes of the equilibrium for large or small changes of fundamentals (e.g. changes in land use, zoning).[2] Second, welfare is endogenously determined as the eigenvalue of the equilibrium system in contrast to standard urban settings where there is a fixed reservation utility (see for example Lucas and Rossi-Hansberg (2003), Ahlfeldt, Redding, Sturm, and Wolf (2012)). That last feature of our theory allows us to use standard mathematical operator theory (see e.g. Polyanin and Manzhirov (2008), Allen, Arkolakis, and Li (2015)) to characterize the changes in welfare as function of changes in policy fundamentals.

Our model constitutes a bridge between standard geography models starting with the seminal work of Krugman (1991) (see, for example, Helpman (1998), Redding and Sturm (2008), Allen and Arkolakis (2014), Redding (2015), Ramondo, Rodríguez-Clare, and Saborío-Rodríguez (2012), Caliendo, Parro, Rossi-Hansberg, and Sarte (2014)) where location of people and production is endogenous and separated by trade costs and urban models where residential and working locations are separated by commuting costs (see Fujita and Ogawa (1982), Lucas and Rossi-Hansberg (2003), Ahlfeldt, Redding, Sturm, and Wolf (2012), and Ioannides (2013) for a comprehensive review). Our work is also related to a large urban literature in urban economics that analyzes optimal spatial policy use in the presence of externalities, reviewed in Glaeser and Gottlieb (2008). Closer to our approach, Turner, Haughwout, and van der Klaauw (2014) evaluate the effect of land use regulation on the value of land use and on welfare. The authors exploit cross-border changes in development, prices, and regulation in regions near municipal borders together with detailed data on the land use and regulations. We complement this work by offering a general equilibrium framework to predict optimal land use.

The remainder of the paper is organized as follows: in the next section, we present the theoretical framework, including both the general model and the quantitative version. In Section 3, we illustrate how the model can be applied empirically by considering the city of

[2]Multiplicity of equilibria is a prevarsive issue in spatial models of economic geography and urban economics. See for example Fujita, Krugman, and Venables (1999) and Fujita and Thisse (2013).

Chicago. Section 4 concludes.

2 Theoretical framework

This section describes our theoretical framework. We assume a standard perfectly competitive goods market with firms that use land and labor to produce goods. The premise of the general model is similar to the canonical Alonso-Mills-Muth model (Alonso et al. (1964), Mills (1967), Muth (1969) see Ioannides (2013) ch. 5 for a description) and in particular we assume individuals have general preferences over good consumption and housing and their income is determined by their working time minus commuting. We extend that framework to allow a choice of leisure and of time spent for productive interactions with other agents, which increase the agent's productivity.

2.1 The general setup

We consider a city consisting of a set of locations $S = \{1, 2, ..., N\}$ that we denote with subscripts. Locations in the city are separated by two types of costs: trade costs and travel costs.[3] In particular to ship a good from $i \in S$ to $j \in S$, quantity $\tau_{ij} \geq 1$ must be shipped in order for one unit to arrive (i.e. τ_{ij} is the iceberg trade cost); similarly, it costs t_{ij} units of time to travel from $i \in S$ to $j \in S$.

In each location, there are buildings for residential and commercial use, whose supply is denoted by $H_j^R : S \to R$ and $H_j^F : S \to R$, $j \in S$, respectively. We consider a general firm technology (e.g. it could be constant or decreasing returns to scale) and make no particular assumptions on the utility function of the consumers. The goods markets are assumed to be perfectly competitive.

We use Δ to denote the set of all firms. The set of goods produced is denoted by Θ and different firms, $\delta \in \Delta$, and we denote the goods produced by a firm δ with $\vartheta(\delta) \in \Theta$. Each firm produces in one location, which we denote as $K(\delta) \in S$. We also denote the set of firms in location $j \in S$ as $\Delta_j = \{\delta | K(\delta) = j\}$, and without the risk of confusion, we also denote the set of firms producing good $\theta \in \Theta$ as $\mathit{\Delta}_\theta = \{\delta | \vartheta(\delta) = \theta\}$. The production function of firm δ is denoted as $y_\delta = f_\delta(l, h)$ where l and h are the effective workers and buildings the firm employees by paying local wage $w_{K(\delta)} : S \to R_{++}$ and rent $r^F_{K(\delta)} : S \to R_{++}$ respectively.

[3] While these two costs are likely to be closely related (and we will assume as much when we get to the empirical portion of the paper), for generality we will allow them to potentially differ.

Firms maximize profit, i.e. solve

$$\max_{l,h} \pi_\delta = p_\delta f_\delta (l, h) - w_{K(\delta)} l - r^F_{K(\delta)} h, \tag{1}$$

where l and h are the labor and land allocations of the firm. Goods can be transported from location $i \in S$ to $j \in S$ by incurring iceberg costs $\tau_{ij} \geq 1$.

Agents are denoted by $\omega \in \Omega$, where Ω is the set of all people, and $\bar{L} \equiv \|\Omega\|$ is the measure of this set. The agents may live and work in different locations. They use residential buildings and purchase consumption goods were they live. The time endowment for agent ω is denoted by $e(\omega) : \Omega \to R_+$, which can be used to commute to work, for productive interactions, to work and to enjoy leisure, and the time allocation devoted to each of these activities is denoted as $e^c(\omega)$, $e^A(\omega)$, $e^w(\omega)$ and $e^l(\omega)$, respectively. An agent choosing to live in $i \in S$ and work in $j \in S$ will have to devote an amount of time equal to the travel time between i and j, t_{ij}, i.e. $e^c(\omega) = t_{ij}$. The agent receives wage income and rental income. The agent's productivity (i.e. efficiency units of labor) is $A_\omega \left(e^A(\omega), i, j\right)$. The wage income is the product of the wage in location j (measured in efficiency units of labor) and the time the agent spends working, $w_j A_\omega e^w(\omega)$. The agent also receives rental income from ownership of residential and commercial buildings and profits from firm ownership. Denote $\{s_\delta(\omega)\}_{\delta\in\Delta}$ as the share of stocks of agent ω in firm δ and $\left\{s^R_k(\omega), s^F_k(\omega)\right\}_{k\in S}$ the corresponding shares on residential and commercial buildings in each location k.

We represent the utility of agent ω who lives in $i \in S$ and works in $j \in S$ by

$$u_\omega \left(e^l(\omega), \{g_\theta(\omega)\}_{\theta\in\Theta}, h^R_i(\omega), i, j\right) \tag{2}$$

where $\{g_\theta(\omega)\}_{\theta\in\Theta}$ are the final amount goods ω consumed and $h^R_i(\omega)$ is the amount of housing (measured in units of area) that agent ω consumes.[4] Notice that the quantity of good $\theta \in \Theta$ that agent ω consumes is:

$$g_\theta(\omega) = \sum_{\delta\in\Delta_\theta} \frac{q_\delta(\omega)}{\tau_{K(\delta)i}}, \tag{3}$$

where $\tau_{K(\delta)i}$ is the trade cost of transporting goods from location $K(\delta)$ to i and $q_\delta(\omega)$ are the amount of goods bought from firm δ. Notice that the utility can also be directly affected by the location of residence and work, e.g. some places may be more attractive than others. In what follows, we assume the utility function is strictly increasing in leisure $e^l(\omega)$, the quantity of each good consumed $\{g_\theta(\omega)\}_{\theta\in\Theta}$, and the quantity of housing $h^R_i(\omega)$.

[4] In the appendix, we allow agents to consume housing in multiple locations.

Agents maximize their utility by choosing where to live, where to work, how to allocate their time across the different activities, and how much to consume, solving:

$$\max_{\{q_\delta(\omega)\},h_i^R(\omega),\left\{e^A(\omega),e^w(\omega),e^l(\omega)\right\},i,j} u_\omega\left(e^l\left(\omega\right),\left\{g_\theta\left(\omega\right)\right\}_{\theta\in\Theta},h_i^R\left(\omega\right),i,j\right)$$

subject to their budget constraint:

$$\sum_{\delta\in\Delta} p_\delta q_\delta\left(\omega\right) + r_i^R h_i^R\left(\omega\right) \leq \sum_{\delta\in\Delta} s_\delta\left(\omega\right)\pi_\delta + \sum_{i\in S} r_i^R s_i^R\left(\omega\right) + \sum_{i\in S} r_i^F s_i^F\left(\omega\right) + w_j A_\omega e^w\left(\omega\right), \quad (4)$$

and their time constraint:

$$e^c\left(\omega\right) + e^A\left(\omega\right) + e^w\left(\omega\right) + e^l\left(\omega\right) \leq e\left(\omega\right), \quad (5)$$

where $A_\omega = A_\omega\left(e^A\left(\omega\right), i, j\right)$ is the effective units of labor agent ω provides and $g_\theta\left(\omega\right)$ as in equation 3.

We denote agent ω's final decision of where to live and work we as functions $I\left(\omega\right)$ and $J\left(\omega\right)$. Also denote $\Omega_i^I = \left\{\omega | I\left(\omega\right) = i\right\}$ the set of people living in location i and $\Omega_j^J = \left\{\omega | J\left(\omega\right) = j\right\}$ the set of agents working in location j. We finally define $L_j^E = \int_{\omega\in\Omega_j^J} A_\omega e^w\left(\omega\right) d\omega$ as the effective labor in location j. Given this setup we discuss the equilibrium in this general model and its efficiency properties.

2.2 Equilibrium in the general model

We now define the equilibrium of this model. For the shake of clarity, we separate the equilibrium conditions into several groups. The first set of equilibrium conditions is that the labor, goods, residential and commercial house markets all clear:

$$\int_{\omega\in\Omega_j^J} A_\omega e^w\left(\omega\right) d\omega = \sum_{\delta\in\Delta_j} l\left(\delta\right) \quad (6)$$

$$\int_{\omega\in\Omega} q_\delta\left(\omega\right) d\omega = f_\delta\left(l\left(\delta\right), h\left(\delta\right)\right) \quad (7)$$

$$\int_{\omega\in\Omega} h_i^R\left(\omega\right) d\omega = H_j^R \quad (8)$$

$$\sum_{\delta\in\Delta_j} h\left(\delta\right) = H_j^F \quad (9)$$

In the first time of equilibrium we consider, we assume the residential and commercial areas in each location are given exogenously. This will prove helpful below in characterizing an equilibrium where the city planner allocates area to different uses through zoning.

Definition 1. Given agent's endowment $\left\{e(\omega), \{s_\delta(\omega)\}_{\delta\in\Delta}, \{s_i^R(\omega), s_i^F(\omega)\}_{i\in S}\right\}$, firms production function $f_\delta(l,h)$ for all $\delta \in \Delta$ and building supply $\{H_i^R, H_i^F\}_{i\in S}$, the vector of consumer choices$\left\{\{q_\delta(\omega)\}_{\delta\in\Delta}, \{h_i^R(\omega)\}_{i\in S}, \{e^A(\omega), e^w(\omega), e^l(\omega)\}, I(\omega), J(\omega)\right\}$, $\{l(\delta), h(\delta)\}$ together with the price variables $\left\{\{r_i^R, r_i^F, w_i\}_{i\in S}, \{p_\delta\}_{\delta\in\Delta}\right\}$ constitutes **a spatial equilibrium with zoning,** equilibrium $\mathcal{F}_1$, if (i) equations (6)-(9) are satisfied; (ii) agents and firms maximize the utility and profit as above.

We now consider an alternative equilibrium, where the market determines the allocation of the residential and commercial buildings by ensuring the commercial and residential rents equalize. The use of the buildings are allowed to be freely changed. The house follows $H_j^R + H_j^F = H_j$. Thus, in equilibrium, the utility (profit) maximization implies $r_j^F = r_j^R$.

Definition 2. The above **spatial equilibrium with zoning,** equilibrium $\mathcal{F}_1$, together with $r_j^F = r_j^R$ and $H_j^R + H_j^F = H_j$ constitutes a **spatial equilibrium without zoning** $\mathcal{F}_2$.

Given these definitions, we now state a theorem that provides sufficient conditions for the spatial equilibrium without zoning to be efficient, $\mathcal{F}_2$:

Theorem 1. *In the spatial equilibrium $\mathcal{F}_2$, if $t_{ij}(\omega)$ and $A_\omega\left(e^A(\omega), i, j\right)$ only depends on ω's individual choice $\left(e^A(\omega), i, j\right)$ (i.e. there is no congestion in commuting and no spillover in production), then if an equilibrium exists, it is efficient.*

Proof. The main idea of the proof is to setup an equivalent economy where we can apply the first fundamental theorem of welfare economics. Details can be found in the appendix. □

The proposition is similar to the result of Konishi (2008) that focuses on an environment with public goods provision (Tiebout equilibrium). Instead, our approach is intentionally tilted to classic urban setups such as extensions of the celebrated von Thunen framework or the more recent approach of Ahlfeldt, Redding, Sturm, and Wolf (2012) and we do not need to impose specific functional form assumptions. In particular, we relax the assumption of a numeraire good, costless trade (we consider a trade environment with an arbitrary number of goods and trade costs) and constant returns to scale with one factor (we consider a general production function with capital and labor and profits that are distributed back to individuals).

2.3 Parametric specification

Whereas the application of the first welfare theorem of economics is straightforward, the characterization of the properties of the equilibrium of this general model is challenging because of the aggregation of agents' decisions.[5] In order to make progress on this front we proceed with a parameterized version of the model and characterize the existence, uniqueness, and comparative statics of a spatial equilibrium in that setup. This parameterized version of the model also proves to be straightforward to bring to the data, as we illustrate in the next section. We will describe the production and agent decisions in turn.

2.3.1 Production

We now assume that all firms in a location produce the same differentiated variety of a good as in Anderson (1979). The production function is assumed to be Cobb-Douglas on land, H_i^F, and effective units of labor, L_i^E, in location i,

$$Y_i = \left(L_i^E\right)^{\alpha} \left(H_i^F\right)^{1-\alpha}, \tag{10}$$

with α being the share of labor and where

$$L_i^E = \int_{\omega \in \Omega_i} A\left(\omega\right) e^w\left(\omega\right) d\omega. \tag{11}$$

Firm optimization (see equation (1)) yields the following expression for wage per efficiency unit of labor:

$$w_i = \alpha p_i \left(L_i^E\right)^{\alpha-1} \left(H_i^F\right)^{1-\alpha}, \tag{12}$$

and for rent per unit of commercial area:

$$r_i^F = \left(1-\alpha\right) p_i \left(L_i^E\right)^{\alpha} \left(H_i^F\right)^{-\alpha}, \tag{13}$$

where $\alpha \in (0, 1]$ and p_i is the price of the good in location i, which from perfect competition is equal to marginal cost:

$$p_i = \frac{w_i^{\alpha} \left(r_i^F\right)^{1-\alpha}}{\alpha^{\alpha} \left(1-\alpha\right)^{1-\alpha}},$$

[5]This result, also known as the Sonnenschein-Mantel-Debreu theorem, is summarized by stating that the aggregate excess demand function resulting from the aggregation of individual agent decisions may not be well behaved. Thus establishing existence, uniqueness, identification, and comparative statics might be particularly hard. See Rizvi et al. (2006).

and the price of the good to the final destination simply incorporates the iceberg trade cost:

$$p_{ij} = p_i \tau_{ij}. \tag{14}$$

2.3.2 Agent's problem

As a specific case of the general problem, we now parametrize the agent's utility function. Consider an agent ω who lives in location $i \in S$ and works in location $j \in S$. We assume the agent ω's utility is a Cobb-Douglas function composed of five parts: the (exogenous) amenity $\overline{u}_i$ where the agent lives; the quantity of time the agent allocates to leisure, $e^l(\omega)$; a CES aggregator of the quantity of goods the agent consumes, $Q_i(\omega)$; the quantity of housing the agent consumes, $h_i^R(\omega)$; and an idiosyncratic preference shock $\upsilon_{ij}(\omega)$ over where she lives and where she works:

$$u_{ij}(\omega) = \bar{u}_i \times e^l(\omega)^{\beta_l} \times Q_i(\omega)^{\beta} \times h_i^R(\omega)^{1-\beta} \times \upsilon_{ij}(\omega),$$

where:

$$Q_i(\omega) = \left(\sum_k q_{ki}(\omega)^{\frac{\sigma-1}{\sigma}} \right)^{\sigma/(\sigma-1)}, \tag{15}$$

and $\sigma > 1$ is the elasticity of substitution and $\beta \in (0, 1]$ is the share of consumption of tradable goods.[6]

Furthermore, we follow Eaton and Kortum (2002) and Ahlfeldt, Redding, Sturm, and Wolf (2012) by assuming that the idiosyncratic preference, $\upsilon_{ij}(\omega)$ follows a Frechet distribution with shape parameter θ, i.e. $Pr[\upsilon_{ij}(\omega) \leq u] \sim e^{-u^{-\theta}}$. We also assume $\upsilon_{ij}(\omega)$ is independent across commuting pairs, i.e. $\upsilon_{ij}(\omega) \perp \upsilon_{mn}(\omega)$ for any $(i, j) \neq (m, n)$.[7]

As in the general model, the agent maximizes her utility subject to both a budget constraint and a time constraint. To specify the budget constraint, we assume there exists a capital pool that accumulates all the rents of commercial and residential building. Then the capital pool redistributes all rent income to agents proportional to their wage income. Due to our Cobb-Douglas production and utility function, in order to exhaust all rent income, the ratio of capital income and wage income turns out to be $\frac{1}{\alpha\beta} - 1$.[8] The budget constraint

[6] While the model attains a non-trivial solution even for $\sigma \in (0, 1)$, we focus on the case where $\sigma > 1$ so that the elasticity of trade flows to trade costs is negative.

[7] See also Redding (2015) and Allen and Arkolakis (2014) for application of this modeling assumption to economic geography models where agents do not commute.

[8] This assumption is made primarily for tractability as it will allow to easily characterize the general equilibrium of the model. Oftentimes firms redistribute part of their profits to their workers, which could explain why commercial rental income is distributed in such a way. Also, past inheritance and other considerations may result to residential income being oftentimes proportional to actual wage income. It worth pointing out

then becomes:

$$\sum_k \tau_{ki} p_{ki}(\omega) q_{ki}(\omega) + r_i^R h_i^R(\omega) \leq \frac{1}{\alpha\beta} w_j A(\omega) e^w,$$

where $A(\omega)$ is agent ω's (endogenous) productivity, which we now discuss.

The time constraint is identical to the general model given by (5) with $e(\omega) = 1$, but we now explicitly specify how an agent increases her productivity by interacting with other persons in the city (e.g. by meeting with other people in different locations in the city, which can be regarded as establishing more connections with business partners, learn new techniques, improving human capital through education, etc). In particular, we assume that agent ω who works in location j has the following productivity:

$$A(\omega) = \bar{A}_j \left[\sum_k \left(\left(L_k^E \right)^\eta l_k^A(\omega)^\mu \right)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{16}$$

where $l_k^A(\omega)$ is how much the agent chooses to interact with persons working in location k, $\mu > 0$ is a parameter governing the productivity of each unit of time spent interacting, $\eta > 0$ is a parameter governing the extent to which the number of persons (measured in efficiency units) working in a location, L_k^E, affects the agent ω's productivity, the parameter ε, $\varepsilon > 0$ and $\varepsilon \neq 1$, governs the substitutability of interactions across space, and $\bar{A}_j$ is the (exogenous) productivity of location j. Our micro-foundation presented in equation (16) shares a common intuition with the recent literature on ideas and growth (see Lucas (2009) and Alvarez, Buera, and Lucas Jr (2013)), and the recent urban literature (see Charlot and Duranton (2004), Glaeser (1999), and Davis and Dingel (2012)) who propose these interactions may serve as a means of transmission of human capital.[9] The time constraint for time spent on improving productivity, given that each trip to location k from agent ω's location of work j requires travel time t_{jk} is given by:

$$\sum_k t_{jk} l_k^A(\omega) \leq e^A(\omega), \tag{17}$$

where we assume for simplicity that agents travel from their place of work to interact with

that in our context this configuration is the only one that will result to an equilibrium where Walras law holds and there no net transfers outside of the system. Alternative configurations that may support efficient allocations can be constructed by assuming that the profits are distributed to individuals that cannot choose their residence and commuting location (see also Monte, Redding, and Rossi-Hansberg (2015) and Fajgelbaum, Morales, Serrato, and Zidar (2015)). We choose the current configuration to avoid taking an explicit stance on the preference of those additional agents in the model and because of the income effects the allocation of these agents may have across different locations.

[9] The literature on human capital spillovers is large. See, for example, Moretti (2004a,b), Combes, Duranton, Gobillon, and Roux (2010) and Ioannides (2013) chapters 5 and 6 for a review.

others.

Solving the agent's problem In the Appendix, we solve the agent's problem: we first maximize the agent's consumption part of the utility function $Q_i(\omega)^\beta \times h_i^R(\omega)^{1-\beta}$, given working time and productivity $A(\omega)$; second, we maximize agent's productivity $A(\omega)$ given time used in interactions; third, we maximize agent's utility by allocating the time into leisure, working and interactions, with the commuting mode as given; finally we maximize agent's utility by choosing the best commuting mode.

Optimal time use From the first three stages we obtain the following equilibrium equations for an agent living in location i and working in location j. The fraction of time spent in leisure is:

$$e_{ij}^l = \frac{\beta_l(1-t_{ij})}{1+\beta_l+\mu}, \tag{18}$$

the fraction of time spent interacting is:

$$e_{ij}^A = \frac{\mu(1-t_{ij})}{1+\beta_l+\mu}, \tag{19}$$

and the fraction of time spent in working is:

$$e_{ij}^w = \frac{1-t_{ij}}{1+\beta_l+\mu}. \tag{20}$$

Given the optimal time use of an agent, she will allocate her time interacting across locations in such a way to make her endogenous productivity to be:

$$A_{ij} = A_j\left(e_{ij}^A\right)^\mu, \tag{21}$$

where:

$$A_j = \bar{A}_j\left[\sum_k \left(t_{jk}^{-\mu} \times \left(L_k^E\right)^\eta\right)^{\frac{\varepsilon-1}{\varepsilon-\mu(\varepsilon-1)}}\right]^{\frac{\varepsilon-\mu(\varepsilon-1)}{\varepsilon-1}} \tag{22}$$

is the *composite productivity* of her workplace in location j and we explicitly restrict $\varepsilon \neq \mu(\varepsilon-1)$.

As in Lucas and Rossi-Hansberg (2003) and Rossi-Hansberg (2005), the endogenous interaction of agents with others implies that that there are external benefits to producers from production done nearby. Our microfoundation implies a spillover function across locations that incorporates as subcases two different specifications in the literature: if $\mu = \frac{1}{\varepsilon-1}$, $\eta = 1$, and commuting times increase exponentially with distance, our formulation matches that of Rossi-Hansberg (2005); conversely if travel times are infinite, then the spillovers are the local spillovers assumed in Allen and Arkolakis (2014).[10] The lateral case is isomorphic to entry externalities present in model with variety entry (see Krugman (1991)) so that our specification generalizes both these previous specifications.

Commuting choice Having characterized the equilibrium allocation of an agent's time conditional on her place of residence and work, we now determine her equilibrium commuting choice. The (indirect) utility function of living in location i and working in location j is:

$$u_{ij}(\omega) = c\bar{u}_i v_{ij}(\omega) \frac{w_j A_j}{d_{ij} P_i^{\beta} \left(r_i^R\right)^{1-\beta}} \tag{23}$$

where P_i is the CES price index and $d_{ij} = \left(\frac{1-t_{ij}}{1+\beta_l+\mu}\right)^{-(1+\mu+\beta_l)}$ is a proportional utility cost that is an explicit function of time required to travel and c is a constant which will not play a role in the subsequent decision and will be ignored henceforth.

In the final stage of the optimization each agent ω will choose the commuting mode which brings the highest utility given equation (23), and the Frechet distribution, so that the share of people commuting from i to j is:

$$\pi_{ij} = \frac{1}{U^{\theta}} \left(\frac{\bar{u}_i A_j w_j}{d_{ij} P_i^{\beta} \left(r_i^R\right)^{1-\beta}}\right)^{\theta}, \tag{24}$$

where

$$U^{\theta} = \sum_{ik} \left(\frac{\bar{u}_i A_k w_k}{d_{ik} P_i^{\beta} \left(r_i^R\right)^{1-\beta}}\right)^{\theta},$$

is the expected utility. We can then determine the number of persons living in i and working in j by multiplying the probability π_{ij} by the measure of agents residing in the city:

$$L_{ij} = \bar{L}\pi_{ij}. \tag{25}$$

Having determined the equilibrium behavior of each agent individually, we now determine the equilibrium economic variables.

[10] Of course, if travel times are infinite, there would be no commuting in this model. In order to allow for commuting and have productivity spillovers be entirely local, we could alternatively assume that the travel time between locations for interactions was infinite, while the travel time between locations for commuting was finite.

2.3.3 Aggregation and equilibrium

To determine the general equilibrium of the model we combine the gravity equations, that determine the flows of goods and labor across space, given factor prices, and determine factor prices, given the market clearing and accounting equations for trade and commuting.

First, given CES preferences over goods, total bilateral sales from location i to location j for the tradable good is given by the familiar gravity equation:

$$X_{ij} = \frac{p_{ij}^{1-\sigma}}{P_j^{1-\sigma}} \beta E_j, \tag{26}$$

with p_{ij} given by

$$p_{ij} = \frac{w_i \tau_{ij}}{\alpha \left(L_i^E\right)^{\alpha-1} \left(H_i^F\right)^{1-\alpha}},$$

where we have used equations (14) and (13). We also define the efficiency units of labor coming from in $i \in S$ used in the production in location $j \in S$ as L_{ij}^E, then the total flows of effective units of labor across space is given by

$$L_{ij}^E = e_{ij}^w A_{ij} L_{ij}, \tag{27}$$

where the individual components in the right hand side are given by equations (20), (21), and (25).

Second, to close the model we consider the equations that define the equilibrium in the goods market and the labor commuting across space and can be used to solve for all the prices, given the gravity equations. The production income in location i can be calculated by summing the sales across destinations:

$$Y_i = \sum_j X_{ij}. \tag{28}$$

Consumers budget balances, which dictates $E_i = \sum_j X_{ji}$. It is straightforward to show that this equation is equivalent to the price index,

$$P_i^{1-\sigma} = \sum_{k=1}^{N} p_{ki}^{1-\sigma}, \tag{29}$$

The income generated from commuters, has to add up to their total spending. Thus, from the agent's budget constraint, total spending can be written as:

$$E_i = \sum_j \frac{1}{\alpha\beta} w_j L_{ij}^E. \tag{30}$$

Finally, the total effective labor in location i can be calculated by summing the effective labor commuting flow across all origins:

$$L_i^E = \sum_j L_{ji}^E. \tag{31}$$

To complete the description of the model we need to include the equation that determines productivity as a function of spatial spillovers, equation (22).

We can now define the equilibrium. Just as in the general model, we first consider a notion of equilibrium where the allocation of land is pre-specified because of regulation (e.g. zoning). This notion will allow us to study the planning problem of choosing the land allocation under the presence of various production externalities.

Definition 3. An equilibrium with zoning (equilibrium $\mathcal{F}_1$), in which variables r_i^R, w_i, P_i, L_i^E, A_i for all i and U are endogenously determined, is characterized by equations (28)-(11), (22), and the aggregate labor clearing condition:

$$\bar{L} = \sum_{i,j} L_{ij}, \tag{32}$$

and where L_{ij}, L_{ij}^E, and X_{ij} are given by equations (25), (27), (26), respectively.

Notice that given these equilibrium solutions we can determine the rent for commercial buildings, r_i^F, using equation (13), the number of residents L_i^R using $L_i^R = \sum_j L_{ij}$. In addition, like in the general model, an equilibrium without zoning can be straightforwardly extended by pooling the residential and commercial land building into a unified rental market.

Definition 4. An equilibrium without zoning (equilibrium $\mathcal{F}_2$), in which variables r_i^R, w_i, P_i, L_i^E, A_i, H_j^F, H_i^R, for all i and U are endogenously determined, is characterized by equilibrium $\mathcal{F}_1$ and the rental market conditions

$$H_j^F + H_j^R = H_j, \tag{33}$$

$$r_j^R = r_j^F. \tag{34}$$

The above completes the derivation of the main elements of the model. We next proceed to the model characterization.

2.4 Equilibrium properties

In this subsection, we characterize the efficiency, existence and uniqueness of the model equilibrium. The first result is a corollary of Theorem 1.

Corollary 1. *If $\eta = 0$, the above equilibrium $\mathcal{F}_2$ is efficient.*

Proof. This result is a a direct application of Theorem 1. See the Theory Appendix for details. □

Recall that if $\eta = 0$ that the number of people working in a location has no effect on the productivity of an agent who interacts with people in that location. Hence, Corollary 1 implies that in the absence of interaction spillovers, a city planner cannot improve efficiency.

By applying theorem 1 of Allen, Arkolakis, and Li (2015), we can prove the following theorem.

Theorem 2. *Consider the above competitive equilibrium with zoning $\mathcal{F}_1$,*

i) An equilibrium always exists.

ii) The equilibrium is (up-to-scale) unique if $\beta \leq \frac{\theta+1}{\theta}\frac{\sigma-1}{\sigma}$ and $\eta = 0$.

iii) Assume no trade costs, i.e. $\tau_{ij} = 1$ for all i, j. Then the equilibrium is unique if $|\eta| \leq \frac{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta}{2\sigma(\theta+1)}$.

Proof. Details are in the appendix. □

The theorem illustrates that the characterization of the properties of the equilibrium of urban models with spatial externalities can be generalized beyond particular examples. Theories that feature technological spillovers across space (see for example Fujita and Ogawa (1982), Lucas and Rossi-Hansberg (2003), Rossi-Hansberg (2005), Fujita and Thisse (2013) chapter 6) usually assume a particular geography (e.g. line or circle) and structure for trade costs or impose a restriction on the diffusion matrix K_{ij}^A, and also typically assumed particular values for the spillover elasticity, $\eta = 0$ or $\eta = 1$.

In contrast, Theorem 2 proves that an equilibrium always exists for any η and for any matrix governing technological diffusion across space.[11] An interesting implication of Theorem 2 is that, when there is no additional restriction on the geography –such as zero trade costs– uniqueness can only be guaranteed when $\eta = 0$, i.e. any spatial productivity spillovers,

[11] Our work is also related to Monte, Redding, and Rossi-Hansberg (2015) who also consider a model of trade and commuting. In their case there are no production externalities, $\eta = 0$. While there are differences in our approaches, our conditions for uniqueness are in principle the same for positive trade costs whereas we also establish conditions for uniqueness in the absence of trade costs if $\eta \neq 0$. Monte, Redding, and Rossi-Hansberg (2015) establish existence for the same parameter restrictions as for uniqueness while Theorem 2 establishes existence for any parameter configuration.

regardless of their strength, may result in multiple equilibria. However, since Corollary 1 implies that the only situation in which zoning may improve efficiency is when $\eta \neq 0$, this means that determining the optimal city planning in the presence of spillovers may necessarily have to contend with possible multiplicities of equilibria. Of course part iii) of the theorem gives us some hope that low η may be enough to guarantee uniqueness. In what follows, we derive a necessary condition for optimality that relies on data from the observed equilibria (which may be one of the many) and we plan to analyze the full city planner problem in the future.

2.5 Analytical Characterization

In order to analyze the main forces at work in our model, we proceed with two analytical examples. In the both these examples we set the trade costs to zero, i.e. $\tau_{ij} \doteq 1$, in order to focus on the urban features of the model, i.e. the commuting costs, and the spatial productivity spillovers.

No trade cost In the first example, we consider the general case with no trade costs. The model in that case retains the spatial features of standard gravity trade models, i.e. there are positive trade flows, albeit the price index is the same across locations. We choose to normalize it to 1, i.e. $P_i = 1$. It turns out that this assumption drastically simplifies the analysis. As a result we can use the first order conditions from the firm and equation (7), combined with equation (11) to write local effective wages, $w_i A_i$, and employment, L_i^E, as log linear functions of local productivity and land for commercial purposes, and a term *multilateral commuting accessibility* term that is a sufficient statistic for the accessibility of a location i to high amenity, low rental price residential locations. In particular, we have:

$$\gamma_1 \ln L_i^E = \theta\sigma \ln A_i + \theta\tilde{\sigma} \ln H_i^F + \sigma \ln \sum_j \bar{u}_j^\theta d_{ji}^{-\theta} \left(r_j^R\right)^{-\theta(1-\beta)} + C_L$$

and

$$\gamma_1 \ln \left(w_i A_i\right) = \sigma \ln A_i + \tilde{\sigma} \ln H_i^F - (\tilde{\sigma}+1) \ln \sum_j \bar{u}_j^\theta d_{ji}^{-\theta} \left(r_j^R\right)^{-\theta(1-\beta)} + C_w,$$

where C_w, C_L are scalars determined by the wage normalization and the aggregate labor clearing constraint, $\tilde{\sigma} = (1-\alpha)(\sigma-1)$, and $\gamma_1 = \theta(\tilde{\sigma}+1) + \sigma$, with $\tilde{\sigma}, \gamma_1 > 0$.

These expressions extend the standard equilibrium equations arising from spatial models (see for example, Kline and Moretti (2014) for an application of the Roback (1982) model, and Allen and Arkolakis (2014), Bartelme (2015), for an application of the gravity model in economic geography) to the case of commuting. A number of comments are in order. First, notice that the effects of productivity and commercial land on employment are positive for both employment and effective wages. Second, higher heterogeneity (lower θ) decreases the elasticity of labor and increases the elasticitiy of effective wages to to these variations in productivity and commercial land supply. Naturally more heterogeneity means that people are less willing to substitute locations with variations in their returns, but also that returns should vary more substantially with fundamentals in equilibrium. Finally, notice that the commuting accessibility term affects employment and local effective wages with a different sign. Higher accessibility implies more employment for a location but also lower effective wages in equilibrium as workers are more willing to work in a location with easy commuting access to residential locations that have high amenity quality (high $\bar{u}_j$) or inexpensive residences (low r_j^R).

To understand the role of the spatial spillover in productivity, we assume further that $\beta = 1$ while maintaining the assumption of no trade costs. It turns out the equilibrium can be represented by one equation (derivations in the appendix.)

$$l_i = \tilde{\lambda} \sum_j \tilde{K}_{ij}^A \left(l_j\right)^{\frac{\eta}{\sigma_1}}$$

where $l_i = \left(L_i^E\right)^{\sigma_1 \tilde{\epsilon}}$, $\sigma_1 = \frac{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta}{\sigma(1+\theta)}$, $\tilde{\lambda} = \left(\lambda \beta^{\frac{\theta}{\sigma}} E^{\frac{\theta}{\sigma}}\right)^{\frac{1}{1+\theta}}$, and $\tilde{K}_{ij}^A = \bar{A}_i t_{ij}^{-\mu\tilde{\epsilon}} \left(\sum_k K_{ik}^{L_E} \left(H_k^F\right)^{\frac{\theta(\alpha-1)}{\sigma}}\right)^{-\frac{1}{1+\theta}}$ is exogenous under $\mathcal{F}_1$ (in which $K_{ij}^{L_E} = \alpha^\theta e_{ji}^w e_{ji}^A \bar{u}_j^\theta \left(H_i^F\right)^{(1-\alpha)\theta} d_{ji}^{-\theta}$).

$$l_i = \tilde{\lambda} \sum_j \tilde{K}_{ij}^A \left(l_j\right)^{\frac{\eta}{\sigma_1}} \tag{35}$$

$\tilde{K}_{ij}^A$ is exogenous under $\mathcal{F}_1$. Besides, one can easily show that uniqueness holds here as long as $|\eta| \leq \frac{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta}{\sigma(1+\theta)}$ which in fact is less restrictive than the one in theorem (2) due to our additional assumptions.

The following are two examples where we can analytically solve the above equation (35). As the examples are in continuous space, all the summation above become integral.[12]

City in the circle: Let S be the circle in $[-\pi, \pi]$. Let's further assume: i) $\bar{A}_i, H_i^F, \bar{u}_i$ are the same for all i; ii) commuting cost is a function of distance. iii) $t_{ij} = \left|\cos\left(\frac{i-j}{2}\right)\right|^{\frac{-2}{\mu\tilde{\epsilon}}}$. Then $\tilde{K}_{ij}^A$ becomes $C\cos^2\left(\frac{x-s}{2}\right)$ (C is a constant). We can show that the solution of l_i is the form of $l_i = Y + \sqrt{Y^2 + C_1}\sin\left(i + C_2\right)$. Y and $C_1 < 0$ are constants and can be solved by substituting the solution back to equation (35). In particular, when $\left|\frac{\eta}{\bar{\sigma}}\right| \leq 1$, $Y^2 + C_1 = 0$.

[12]Note that formally our analysis does not apply to a continuum of regions because of the discrete choice problem. However, you can think of the examples below as limits as the number of regions tends to a large number.

And C_2 can be any constant which is a result of the symmetry of circle. In other words the solution is the symmetric solution if we stay in the region where equilibrium is unique. In contrast, when $\frac{\eta}{\tilde{\sigma}} > 1$ simulations indicate the existence of a multiple equilibria were more concentration can happen in any point on the line $(Y^2 + C_1 > 0)$

City in the infinite line: Let S be the circle in $[-\infty, \infty]$. Also let's further assume: i) $\bar{A}_i, H_i^F, \bar{u}_i$ are the same for all i; ii) commuting cost is a function of distance i.e. $d_{ij} = f(|i-j|)$; iii) $t_{ij} = \exp\left(\tau (i-j)^2\right)$ where τ is used to capture the unit. Then $\tilde{K}_{ij}^A$ becomes $C \exp\left(-\mu\tilde{\epsilon}\tau (i-j)^2\right)$ (C is a constant). Then we can show that when $\eta > \tilde{\sigma}$ the solution of l_i is the form of $l_i = C_1 \exp\left(-\mu\tilde{\epsilon}\tau \frac{\eta-\tilde{\sigma}}{\tilde{\sigma}} (x + C_2)^2\right) Y$. $C_1 > 0$ is a constant and can be solved by substituting the solution back to equation (35). And C_2 can be any constant which is a result of the symmetry of infinite line. When $\eta \le \tilde{\sigma} \le 1$, the agglomeration force is smaller than dispersion force. Population is evenly distributed in the infinite line. There is no specific function to represent the solution.

2.6 City Planner

In Theorem 1 we showed that in the absence of any externalities the equilibrium of the general model is efficient. In the presence of production externalities, however, even in the parametrized model the competitive equilibrium is not necessarily efficient. In that case there is room for policy intervention, for example land-use reallocation through zoning. For that purpose we now characterize the zoning problem, where a central planner can decide on the allocation of land for residential and commercial use. To do this, we consider a hypothetical planner that changes $\left\{H_i^F, H_i^R\right\}$ (taking as given the total land available, $H_i = H_i^F + H_i^R$) to achieve a higher social welfare U. To proceed we follow Dekle, Eaton, and Kortum (2008)'s 'hat' algebra, we can rewrite the equilibrium conditions (55)-(59) in changes ,

$$\hat{U}^{-\theta} \sum_j \pi_{ij}^w \left(\hat{w}_j \hat{A}_j\right)^{1+\theta} \hat{P}_i^{-\theta\beta} \left(\hat{r}_i^R\right)^{-\theta(1-\beta)} = \hat{H}_i^R \hat{r}_i^R \tag{36}$$

$$\hat{U}^{-\theta} \sum_j \pi_{ij}^L \hat{w}_j^\theta \hat{A}_j^{1+\theta} \hat{P}_i^{-\theta\beta} \left(\hat{r}_i^R\right)^{-\theta(1-\beta)} = \hat{L}_i^E \tag{37}$$

$$\sum_j \pi_{ij}^Y \hat{H}_j^R \hat{r}_j^R \hat{p}_i^{1-\sigma} \hat{P}_j^{\sigma-1} = \hat{p}_i \left(\hat{L}_i^E\right)^\alpha \left(\hat{H}_i^F\right)^{1-\alpha} \tag{38}$$

$$\sum_j \pi_{ij}^E \hat{p}_j^{1-\sigma} = \hat{P}_i^{1-\sigma} \tag{39}$$

$$\sum_j \pi_{ij}^A \left(\hat{L}_j^E\right)^{\eta\tilde{\epsilon}} = \hat{A}_i^{\tilde{\epsilon}} \tag{40}$$

and also the change form of labor normalization condition 32

$$\sum_{i,j} \pi_{ij} \left(\hat{w}_j \hat{A}_j\right)^\theta \hat{P}_i^{-\theta\beta} \left(\hat{r}_i^R\right)^{-\theta(1-\beta)} = \hat{U}^\theta \tag{41}$$

where $\hat{w}_j = \hat{p}_i \left(\hat{L}_i^E\right)^{\alpha-1} \left(\hat{H}_i^R\right)^{1-\alpha}$ and we additionally define the following share matrices: $\pi_{ij}^w = \frac{\frac{1}{\alpha\beta} w_j A_{ij} e_{ij}^w L_{ij}}{E_i}$ is the income share in location i by people commuting to j; $\pi_{ij}^L = \frac{A_{ij} e_{ij}^w L_{ij}}{L_i^e}$ the effective labor share in location i from j; $\pi_{ij}^Y = \frac{\pi_{ij}^E \beta E_j}{Y_i}$ is the sales share of goods produced in location i from j;$\pi_{ij}^E = \frac{(\tau_{ji} p_j)^{1-\sigma}}{P_i^{\sigma-1}}$ is the spending share of goods in location i to goods produced in j; $\pi_{ij}^A = \frac{\bar{A}_i^{\tilde{\epsilon}} t_{ij}^{-\mu\tilde{\epsilon}} \left(L_j^E\right)^{\eta\tilde{\epsilon}}}{A_i^{\tilde{\epsilon}}} = \frac{t_{ij}^{-\mu\tilde{\epsilon}} \left(L_j^E\right)^{\eta\tilde{\epsilon}}}{\sum_k t_{ik}^{-\mu\tilde{\epsilon}} \left(L_k^E\right)^{\eta\tilde{\epsilon}}}$ is the productivity spillover from location j; π_{ij} is the labor share in location i from j as in equation 14.

The characterization of the "optimal city structure" raises a number of challenges. First, the large dimensionality of the parameter space – the fraction of land allocated residential and commercial in each location, $[0,1]^N$ – prevents a brute force optimization procedure from being feasible. Second, there are a large number of unobservable factors that determine the equilibrium (e.g. the productivity in each location), so it is not clear that observable data is sufficient to characterize the necessary conditions for the planning problem. Finally, given the results above, we know there may exist multiple equilibria which the planner would have to contend. The following proposition illustrates that what seems at first glance a Herculian task is actually feasible in our setup;

Proposition 1. *Consider the system of equations in changes, (36)-(41). Assume that the planner can control zoning i.e* $\left\{H_i^F, H_i^R\right\}$*, under the constraint* $H_i^F + H_i^R = H_i$ *in each location* i*. Then*

(i) (Global comparative statics) Given shares $\left\{\pi^w, \pi^L, \pi^Y, \pi^E, \pi^A\right\}$ *, there exists a unique solution of the changes in social welfare* $\hat{U}$*, and changes* $\hat{r}_i^R$*,* $\hat{p}_i$*,* $\hat{P}_i$*,* $\hat{A}_i$*,* $\hat{L}_i^e$ *if* $\beta \leq \frac{\theta+1}{\theta}\frac{\sigma-1}{\sigma}$ *and* $\eta = 0$*.*

(ii) (Local comparative statics) Even if $\eta \neq 0$*, the elasticities of city welfare with respect to an increase in the endowment of residential and commercial area, respectively, around the*

observed equilibrium are:

$$\frac{\partial \ln U}{\partial \ln H_{Ri}} = \left(\mathbf{M}^{+}\mathbf{B}\right)_{5N+1,i}$$
$$\frac{\partial \ln U}{\partial \ln H_{Fi}} = \left(\mathbf{M}^{+}\mathbf{B}\right)_{5N+1,N+i},$$

where:

$$\mathbf{M}^{+} = \begin{pmatrix} -\left(1+\theta\left(1-\beta\right)\right)\mathbf{I} & \left(1+\theta\right)\boldsymbol{\pi^{w}} & -\theta\beta\mathbf{I} & \left(1+\theta\right)\pi^{\mathbf{w}} & -\left(1+\theta\right)\left(1-\alpha\right)\boldsymbol{\pi^{w}} & -\theta\vec{1} \\ -\theta\left(1-\beta\right)\pi^{\mathbf{L}} & \theta\mathbf{I} & -\theta\beta\pi^{\mathbf{L}} & \left(1+\theta\right)\mathbf{I} & -\left[1+\theta\left(1-\alpha\right)\right]\mathbf{I} & -\theta\vec{1} \\ \pi^{\mathbf{Y}} & -\sigma\mathbf{I} & \left(\sigma-1\right)\pi^{\mathbf{Y}} & 0 & -\alpha\mathbf{I} & \vec{0} \\ 0 & \left(1-\sigma\right)\pi^{E} & -\left(1-\sigma\right)\mathbf{I} & 0 & 0 & \vec{0} \\ 0 & 0 & 0 & -\tilde{\epsilon}\mathbf{I} & \eta\tilde{\epsilon}\boldsymbol{\pi}^{\mathbf{A}} & \vec{0} \\ -\theta\left(1-\beta\right)\vec{\pi}_{r} & \theta\vec{\pi}_{c} & -\theta\beta\mathbf{I}\vec{\pi}_{r} & \theta\tilde{\pi}_{\mathbf{c}} & -\theta\left(1-\alpha\right)\vec{\pi}_{c} & -\theta\vec{1} \end{pmatrix}^{+}$$

$$\mathbf{B} = \begin{pmatrix} I & -\left(1+\theta\right)\left(1-\alpha\right)\pi^{\mathbf{w}} \\ 0 & -\theta\left(1-\alpha\right)I \\ -\pi^{\mathbf{Y}} & \left(1-\alpha\right)I \\ 0 & 0 \\ 0 & 0 \\ 0 & -\theta\left(1-\alpha\right)\vec{\pi}_{c} \end{pmatrix}$$

where the superscript + denotes the Moore Penrose pseudoinverse of the matrix and both $\vec{\pi}_r$ and $\vec{\pi}_c$ are row vectors $\left(\vec{\pi}_r\right)_i = \sum_j \frac{L_{ij}}{\bar{L}}$ $\left(\vec{\pi}_c\right)_i = \sum_j \frac{L_{ji}}{\bar{L}}$.

(iii) (Optimal city structure) A necessary condition of the optimal city zoning plan is, for all $i \in S$ $H_i^R \left(\mathbf{M}^{+}\mathbf{B}\right)_{5N+1,N+i} = H_i^F \left(\mathbf{M}^{+}\mathbf{B}\right)_{5N+1,i}$.

Proof. Part (i) follows immediately from Theorem 2, as the mathematical structure of the equilibrium in changes is the same as the mathematical structure of the equilibrium in levels (with the exogenous kernels replaced with data). Part (ii) is a direct implication of the implicit function theorem to the vector function constructed by "stacking" equations (36)-(41) and differentiating (the log of) utility with respect to (the log of) H_{Fi} and H_{Ri}. Part (iii) then follows directly from part (ii), as at the optimal zoning level, it cannot be the case that reallocating land from residential to commercial purposes can either increase or decrease the utility of the city, i.e. $\frac{\partial \ln U}{\partial \ln H_{Fi}}\frac{1}{H_{Fi}} = \frac{\partial \ln U}{\partial \ln H_{Ri}}\frac{1}{H_{Ri}}$. □

The first part of the proposition establishes a sufficient set of data that we need in order to consider the equilibrium of the model in changes, as in Dekle, Eaton, and Kortum (2008). It also establishes that the sufficient condition for the global comparative static to be well defined is the same as the sufficient condition for the equilibrium to be unique, i.e. $\eta = 0$. The second part of the proposition allows us to simultaneously calculate how a change in either the endowment of residential or commercial area in any location affects the welfare of the city. It holds even in the presence of spillovers by considering a local shock around the observed equilibrium and depends only on observed data and the set of scalars determining the strength of general equilibrium forces, The third part of the proposition (which follows immediately from the second part and the first order conditions of the social planner) provides necessary conditions that hold at the optimal city zoning plan.

2.7 Examples

TBD

In the next section, we illustrate how we can use this theoretical framework and the results of Proposition 1 to examine the optimal city structure of Chicago.

3 Optimal City Structure: Chicago

In this section, we apply the theoretical framework above to study the optimal structure of the city of Chicago. We first discuss the parameter estimation and use detailed data for the city of Chicago to explore beneficial alternative zoning policies.

3.1 Identification and Estimation

Suppose for all $i \in S$ we observe the model parameters β_l, μ, β and α, residential square footage $\left\{H_i^R\right\}$, commercial square footage $\left\{H_i^F\right\}$, for all bilateral pairs, $i, j \in S$ we observe bilateral commuting flows (with error) $\left\{L_{ij}^o\right\}$ and the travel time between the two locations $\{t_{ij}\}$. We would like to recover the unobserved model parameter θ and the (composite) productivity $\{A_i\}$ and amenity $\{u_i\}$ in each location. We describe how we recover the model parameter θ, the unobserved endogenous incomes and expenditures, and the productivities and amenities.

Estimation of preference heterogeneity θ To identify the preference heterogeneity parameter, we rely on equation (25), which implies the number of workers living in location i and commuting to location j is:

$$L_{ij} = (1 + \beta_l + \mu)^{\theta(1+\mu+\beta_l)} \frac{\bar{L}}{U^\theta} (e - t_{ij})^{\theta(1+\mu+\beta_l)} \left(\frac{\bar{u}_i A_j w_j}{P_i^\beta \left(r_i^R\right)^{1-\beta}} \right)^\theta . \tag{42}$$

Recalling that we observe bilateral commuting flows with error (which we assume is classical), we can take logs of this expression, yielding:

$$\ln L_{ij}^{o} = \rho \ln (e - t_{ij}) + \gamma_i + \delta_j + \varepsilon_{ij}, \tag{43}$$

where $\rho \equiv \theta (1 + \mu + \beta_l)$, $\gamma_i \equiv \theta \ln \left(P_i^{\beta} r_{Ri}^{1-\beta} \right)$, and $\delta_j \equiv \theta \ln (A_j w_j)$. As long as the measurement error is uncorrelated with the bilateral travel times, we can recover ρ from (43), which given knowledge of μ and β_l allows us to recover θ. Intuitively, the more heterogeneous individuals' preferences are (the lower the θ), the less responsive commuting patterns are to observed travel time.

Estimation of income and expenditure. Given estimates of θ, we can then recover the wage per unit labor (note that this the wage per efficiency unit multiplied by the productivity) from the destination fixed effect:

$$\widetilde{A_j w_j} = \exp \frac{\tilde{\delta}_j}{\tilde{\theta}},$$

where the tildes denote an estimate and we pin down the scale by normalizing the wage in one location, i.e. $w_1 = \frac{1}{A_1}$ so that $\delta_1 = 0$.

Combining equations (20), (21), (27) with (30), we can then recover an estimate of expenditure in each location using observed travel time, estimated wages (per unit labor), and the predicted commuting flows from regression (43):

$$\tilde{E}_i = \frac{1}{\alpha\beta} \left(\frac{\mu}{1 + \beta_l + \mu} \right)^{\mu} \sum_j (e - t_{ij})^{1+\mu} \tilde{L}_{ij} \widetilde{A_j w_j}. \tag{44}$$

Similarly, we can use the assumption that the production function is Cobb-Douglas to recover the value of output in each location:

$$\tilde{Y}_j = \frac{1}{\alpha\mu} \left(\frac{\mu}{1 + \beta_l + \mu} \right)^{1+\mu} \widetilde{w_j A_j} \sum_i (e - t_{ij})^{1+\mu} \tilde{L}_{ij}, \tag{45}$$

and, with both observed expenditure and recovered output, we can immediately recover the residential and commercial rental rates in each location using the Cobb-Douglas utility function of the residents and the Cobb-Douglas production function of firms, respectively:

$$\tilde{r}_i^R = (1 - \beta) \frac{\tilde{E}_i}{H_i^F}$$
$$\tilde{r}_i^F = (1 - \alpha) \frac{\tilde{Y}_i}{H_i^F}.$$

In what follows, we omit the hat notation for clarity, but the reader should keep in mind that income, expenditure, and rents are all estimated from the structure of the model.

Estimation of productivities and amenities Having recovered the model parameters, output, and the rental rates, we can now recover the amenities and productivities. From equation (42), the number of workers both living and working in location i is:

$$L_{ii} = (1+\beta_l+\mu)^{\theta(1+\mu+\beta_l)} \frac{\bar{L}}{U^{\theta}} e^{-\theta(1+\mu+\beta_l)} \left(\frac{\bar{u}_i A_i w_i}{P_i^{\beta} \left(r_i^R\right)^{1-\beta}} \right)^{\theta}, \tag{46}$$

so that we can write the amenity of living in location i as:

$$\begin{aligned} \bar{u}_i^{\frac{1-\sigma}{\beta}} &= (1+\beta_l+\mu)^{\left(\frac{\sigma-1}{\beta}\right)(1+\mu+\beta_l)} \left(\frac{U}{\bar{L}^{\frac{1}{\theta}}} \right)^{\frac{1-\sigma}{\beta}} e^{\left(\frac{1-\sigma}{\beta}\right)(1+\mu+\beta_l)} \left(r_i^R\right)^{\left(\frac{1-\sigma}{\beta}\right)1-\beta} \left(A_i w_i\right)^{\frac{\sigma-1}{\beta}} L_{ii}^{\frac{1-\sigma}{\theta\beta}} P_i^{1-\sigma} \iff \\ \bar{u}_i^{\frac{1-\sigma}{\beta}} &= \lambda \sum_j K_{ji} A_j^{(\sigma-1)\alpha}, \end{aligned} \tag{47}$$

where the second line used the fact that the price index can be written as

$$P_i^{1-\sigma} = \sum_j \tau_{ji}^{1-\sigma} \left(\frac{w_j^{\alpha} \left(r_j^F\right)^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \right)^{1-\sigma},$$

λ is an endogenous constant,[13] and

$$K_{ji} \equiv \tau_{ji}^{1-\sigma} \left(\frac{A_i w_i}{L_{ii}^{\frac{1}{\theta}} \left(r_i^R\right)^{1-\beta}} \right)^{\frac{\sigma-1}{\beta}} \left(\left(A_j w_j\right)^{\alpha} \left(r_j^F\right)^{1-\alpha} \right)^{1-\sigma},$$

is a kernel that depends only on observables.

Equation (47) is one of two equations that allows us to identify productivities and amenities. The other comes from the production side. In equilibrium, total income in a location is equal to total sales, i.e. $Y_i = \sum_j X_{ij}$. From equation (26), we can write this relationship as:

$$Y_i = \beta \left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{\sigma-1} \sum_j \tau_{ij}^{1-\sigma} \left(\left(\frac{w_i A_i}{A_i} \right)^{\alpha} \left(r_i^F\right)^{1-\alpha} \right)^{1-\sigma} E_j P_j^{\sigma-1}. \tag{48}$$

[13] In particular, $\lambda \equiv (1+\beta_l+\mu)^{\left(\frac{\sigma-1}{\beta}\right)(1+\mu+\beta_l)} \left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{\sigma-1} \left(\frac{U}{\bar{L}^{\frac{1}{\theta}}}\right)^{\frac{1-\sigma}{\beta}} e^{\left(\frac{1-\sigma}{\beta}\right)(1+\mu+\beta_l)}$.

From equation (46), we have that the price index can be written as:

$$P_i = (1+\beta_l+\mu)^{\frac{(1+\mu+\beta_l)}{\beta}} \frac{\bar{L}^{\frac{1}{\beta\theta}}}{U^{\frac{1}{\beta}}} e^{-\frac{1+\mu+\beta_l}{\beta}} \left(\frac{\bar{u}_i A_i w_i}{(r_i^R)^{1-\beta}}\right)^{\frac{1}{\beta}} L_{ii}^{-\frac{1}{\beta\theta}}. \tag{49}$$

Combining equations (48) and (49) yields:

$$Y_i A_i^{\alpha(1-\sigma)} = \lambda\beta \sum_j K_{ij} \bar{u}_j^{\frac{\sigma-1}{\beta}} E_j. \tag{50}$$

where λ and K_{ij} are defined above.

Equations (47) and (50) can be solved simultaneously to yield the unique (to scale) distribution of composite productivities and amenities. We formalize this statement in the following proposition:

Proposition 2. *Consider an equilibrium with zoning. For any given commuting flows L_{ij}, expenditure E_i, income Y_i, residential stock H_R, commercial stock H_F, bilateral commuting costs t_{ij}, trade costs τ_{ij}, and scalar model parameters, there exists a unique (to-scale) set of productivities, A, and amenities, u, satisfying equations (47) and (50).*

Proof. Equations (47) and (50) can be rewritten as:

$$x_i = \lambda \sum_j K_{ji} y_j^{-1}$$
$$y_i = \lambda \sum_j \beta \frac{E_j}{Y_i} K_{ij} x_i^{-1},$$

where $x_i \equiv \bar{u}_i^{\frac{1-\sigma}{\beta}}$ and $y_i \equiv A_i^{\alpha(1-\sigma)}$ and the kernels K_{ji} and $\beta\frac{E_j}{Y_i}K_{ij}$ are observed. Note that the sufficient condition for existence and uniqueness of a (to-scale) solution given by Allen, Arkolakis, and Li (2015) – that the spectral radius of the matrix of the absolute value of exponents is less than or equal to one – is satisfied here, as the absolute value of the matrix of exponents is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has spectral radius of one. □

3.2 Data

In this section, we briefly describe the data used in the empirical application below.

Transportation infrastructure Data on the location of the "Metra" commuter rail lines and stops, the "El" subway lines and stops, bus lines and stops, and roads are available in

GIS format from Chicago (2015). Figure 1 depicts the transportation infrastructure.

With this data, we constructed a public transportation and a private transportation "speed image", where each map pixel was assigned a relative speed across which it could be traversed based on the infrastructure available at the pixel.[14] We then applied the Fast Marching Method algorithm (see Sethian (1999)) to calculate the time it would take to travel between any of the 2,298 census blocks groups in Cook County, IL, yielding 5.28 million bilateral travel times.

To determine the absolute travel time speed, for 500 randomly chosen bilateral pairs within the city limits of Chicago, we query Google Maps for the travel times, project their estimated travel times on our own, and scale our estimated travel times accordingly. Table 1 presents the results of such a procedure. There are two things to note: first, our estimated travel times are strongly correlated with the Google Maps estimates, with an R-squared of 0.86 for travel by car and 0.8 for travel by public transportation; second, conditional on our estimated travel times, straight line distance offers very little additional predictive power (raising the R-squared to 0.87 and 0.81, respectively).

Finally, we calculate the total travel time between any two locations by assuming that each individual chooses the mode which offers them the shortest commute time up to an extreme value idiosyncratic error so that

$$t_{ij} = -\frac{1}{\lambda} \ln \left(0.5 \times \left(\exp\left(-\lambda t_{ij}^{\text{car}}\right) + \exp\left(-\lambda t_{ij}^{\text{public}}\right)\right)\right),$$

where we choose $\lambda = 0.115$ so that 12% of workers choose to travel to work using public transportation, which is the consistent with the American Community Survey (see McKenzie and Rapino (2011)).

Zoning and buildings Chicago (2015) also provide information on the location, number of stories, and footprint for each of the 820,944 buildings in Chicago, from which we can construct the (approximate) square footage of each building. The left panel of Figure 2 presents a map of the buildings. We then combine this data with how each building is zoned (see the right panel of Figure 2) and aggregate up to Census block to determine the total square feet of residential and commercial space in each Census block.[15] Figure 3 depicts the

[14] In particular, we normalized the speed of travel via water and through buildings to one. Relative to this speed, we assumed for private transportation, one could walk three more quickly, drive through surface streets 25 times more quickly, drive on expressways 40 times more quickly, and drive on interstates 50 times more quickly. For public transportation, we assumed that buses and subways traveled 30 times more quickly and the commuter rail traveled 45 times more quickly. To add greater verisimilitude, we assumed that all three had to slow down when traveling through a stop, and we forced passengers to enter the network at a stop by constructing high travel cost "walls" surrounding the lines at non-stop locations.

[15] In reality, in Chicago there are twelve different zone types, and within zone type there are often many

distribution of area allocated to residential and commercial use in Chicago.

Commuting Flows The 2010 Longitudinal Employer-Household Dynamics Origin-Destination Employment Statistics (LODES) dataset from Census (2010) reports the bilateral flow of commuters from residence to workplace at the U.S. Census block level. The LODES dataset is constructed from unemployment insurance filings of businesses and cover approximately 95% of workers (see e.g. Graham, Kutzbach, and McKenzie (2014)). We aggregate to the Census block group level and only include Census block groups with both residents and workers (and positive area allocated to both residential and commercial uses), leaving 2,036 locations throughout Chicago.

One difficulty with using the LODES data set is that because there are roughly twice the number of cells in the bilateral commuting matrix than the number of people living in Chicago, so many elements of the bilateral commuting matrix contains either a small number of persons or no persons at all. As a result, we treat the observed commuting matrix as one measured with error (see above).

3.3 Estimation results

Calibrated parameters We first calibrate a number of parameters using information that we directly observe in the data. To calibrate μ, β_l we use equations (18) to (20) that represent the share of time, net of time spent in commuting, allocated to leisure, interactions to improve productivity, and working. Dividing equations (18) and (20) by equation (19) we directly obtain β_l and μ. Thus, if we could obtain estimates of these times and commuting times we can directly calibrate those parameters. We construct estimates for these times using Census (2011) and BLS (2013) and using those we obtain $\beta_l = 7.33$ and $\mu = 4.12$. We provide details in the appendix B.

For the parameter α we do not have a good sense of its value and we set it for the time being to 0.75. We plan to consider detailed data on the share of the costs of commercial building use in the future. For β we use the average fraction of observed expenditure on rent from American Community Survey (MPC, 2011) for Chicago (38%), yielding $\beta = 0.62$. We also calibrate the elasticity of trade, σ, to the value estimated by Eaton and Kortum (2002), following Allen and Arkolakis (2014). We set $\varepsilon = 1$ so that productivity aggregates across locations in a Cobb-Douglas matter and choose $\eta = 0.02$ to match the estimates of Charlot

different particular codes, so the mapping to the residential/commercial binary is necessarily imprecise. However, since we do have the number of stories of each building, we can incorporate such subtleties as Zone 2 ("Neighborhood commercial with dwelling units above ground") by allocating the first floor to commercial and all higher floors to residential.

and Duranton (2004) for the extent of communication externalities in cities.

Estimation of preference heterogeneity θ Estimation of equation (43) yields a coefficient of 8.73 (with a t-statistic 125), which, given the calibrated values of β_l and μ above, imply a $\theta = 0.71$. This low value of θ indicates substantial heterogeneity in preferences across locations, which will imply there will be limited responsiveness of individuals changing where they live or where they work to changes in the structure of city.

Income and expenditure Figure 4 depicts the distribution of income and expenditure across Chicago calculated by applying equations (44) and (45) to the regression in equation (43). As is evident, income and expenditure is strongly positively correlated, with residents and firms located in the central business district have the highest expenditure and income, respectively.

We should emphasize that the income and expenditure depicted in Figure 4 (and used in the counterfactual analysis) are inferred from travel times and commuting patterns. While it is difficult to directly observe firm level income at such a fine level of geographic detail, we can compare our estimates of variation in expenditure of residents to reported per capita income at the Census block group level from (MPC, 2011). We find that the estimates are positively correlated with a correlation of 0.32 (0.2 correlation between estimated log expenditure and observed log per capita income).

Composite productivities and amenities Given estimates of income and expenditure, we can finally apply the result of Proposition 2 to identify the (total) productivity and amenity of every location in Chicago. Figure 5 depicts the resulting spatial distribution. While both the productivity and amenity of the central business district is estimated to be high, the productivities and amenities elsewhere are negatively correlated. For example, the amenity value of living along the coast of Lake Michigan is estimated to be high, but the productivity of businesses located along the coast generally tend to be low.

3.4 Optimal zoning in Chicago

Given the estimates from the previous section, we finally examine how small changes in the zoning of Chicago would affect the welfare of residents and what this implies for the optimal city structure.

We first examine what the effect would be on city wide welfare if there was a slight increase in the residential or commercial area of a particular location in the city. While conventional techniques would allow us to analyze an increase in a particular location (by recalculating the equilibrium distribution of economic activity after a small change to the endowment of area in a location), we instead rely on the results of Proposition 1 to do this for all locations simultaneously.

Figure 6 illustrates the distribution of welfare elasticities across the city. Not surprisingly, the welfare effects of expanding the endowment of both commercial and residential area have the largest positive effects on welfare near the center of the city, where the commuting and interacting costs are relatively low.

Once we have calculated both the welfare elasticity to expanding both the commercial and residential areas, we can compare the elasticities to infer what the optimal zoning policy is. That is, if $\frac{\partial \ln U}{\partial \ln H_{Ri}} \frac{1}{H_{Ri}} > \frac{\partial \ln U}{\partial \ln H_{Fi}} \frac{1}{H_{Fi}}$, then re-zoning some of the commercial area in a location to residential be welfare improving. Figure 7 illustrates, for every Census block group in Chicago, whether the welfare of the city would increase if more of the area was re-zoned residential or commercial. As is evident, the model suggests that it would be welfare improving if Chicago re-zoned commercial areas throughout the central business district as residential, whereas it would be welfare improving if Chicago re-zoned much of the residential areas outside the business district to allow for more commercial use.

Two caveats are in order for this exercise. First, the re-zoning exercise considered is for a small (infinitesimal) change; while all general equilibrium effects are accounted for (to a first order), higher order general equilibrium effects may be important for large changes. Second, as is evident from the functional form of the $\mathbf{M}^+$ matrix in Proposition 1, the counterfactual results depend importantly on the values of the structural parameters chosen. In particular, altering the strength of productivity spillovers η and the substitutability of interacting across space ε play an important role in determining the optimal structure of the city. Indeed, as Theorem 1 makes clear, in the absence of spillovers, there ought not be any zoning at all.

4 Conclusion

TBD

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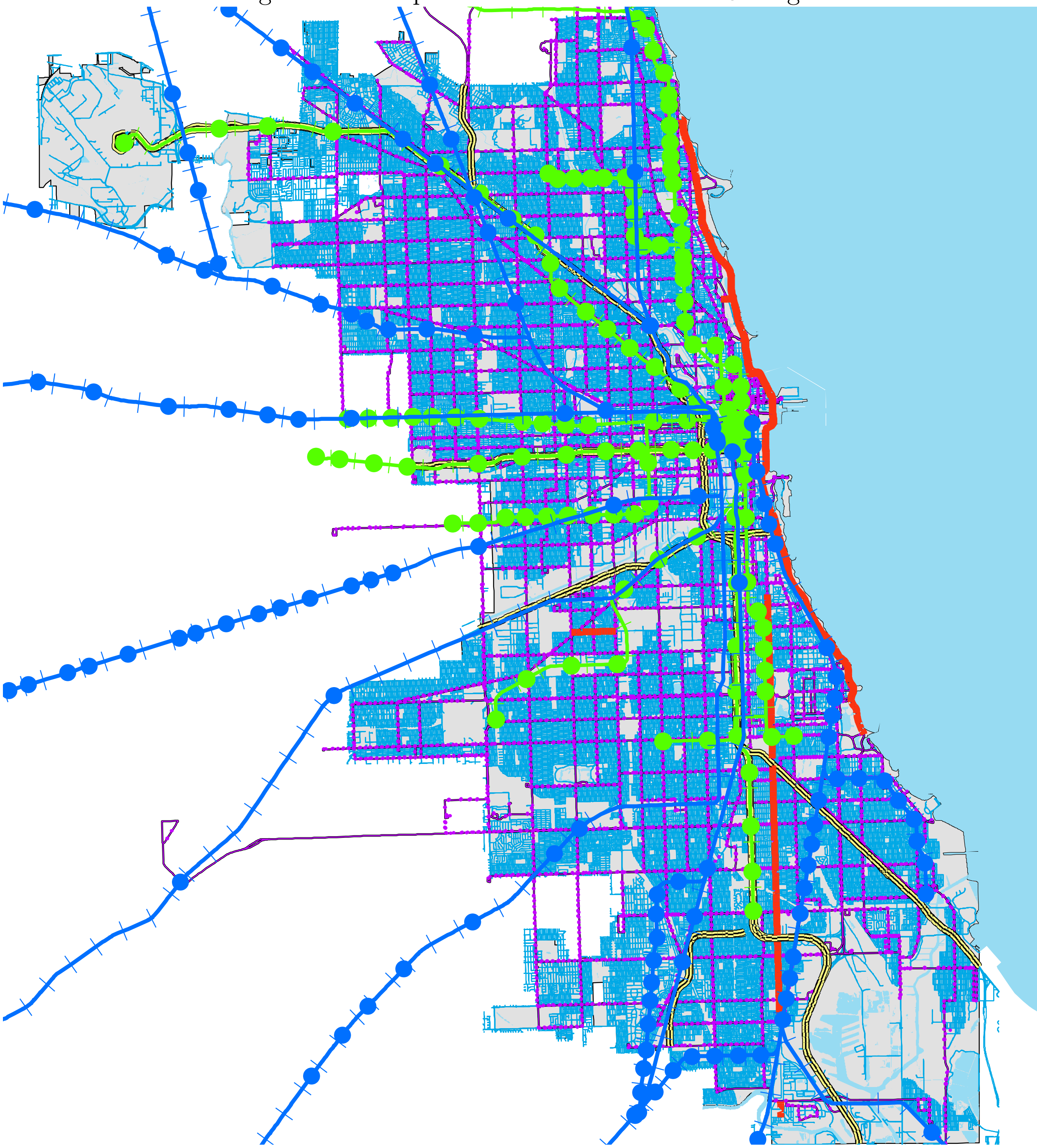
Tables and figures

Table 1: Travel time: Fast Marching Method vs. Google Maps

	Google travel time by car		Google travel time by public transit	
	(1)	(2)	(3)	(4)
FMM estimated distance by car	1.000***	1.253***		
	(0.018)	(0.049)		
FMM estimated distance by public transit			1.000***	0.837***
			(0.022)	(0.056)
Straightline distance		-0.508***		0.797***
		(0.091)		(0.252)
Observations	500	500	500	500
R-squared	0.858	0.866	0.804	0.808

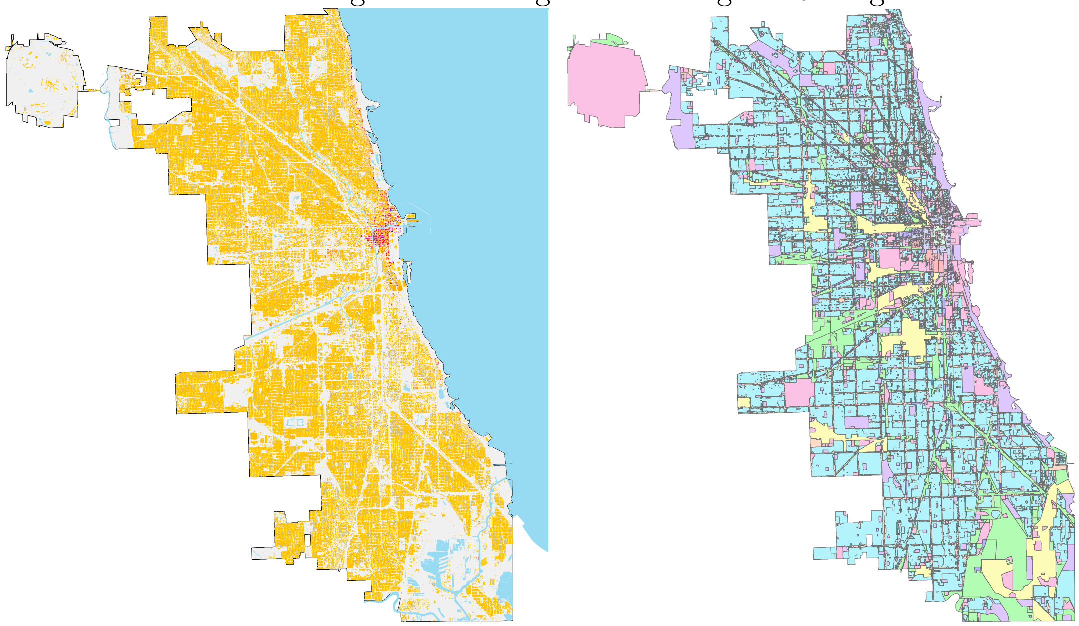
Notes: The dependent variable is indicated above each column. The observations are 500 randomly selected bilateral pairs. Stars indicate statistical significance: * $p<.10$ ** $p<.05$ *** $p<.01$.

Figure 1: Transportation infrastructure in Chicago



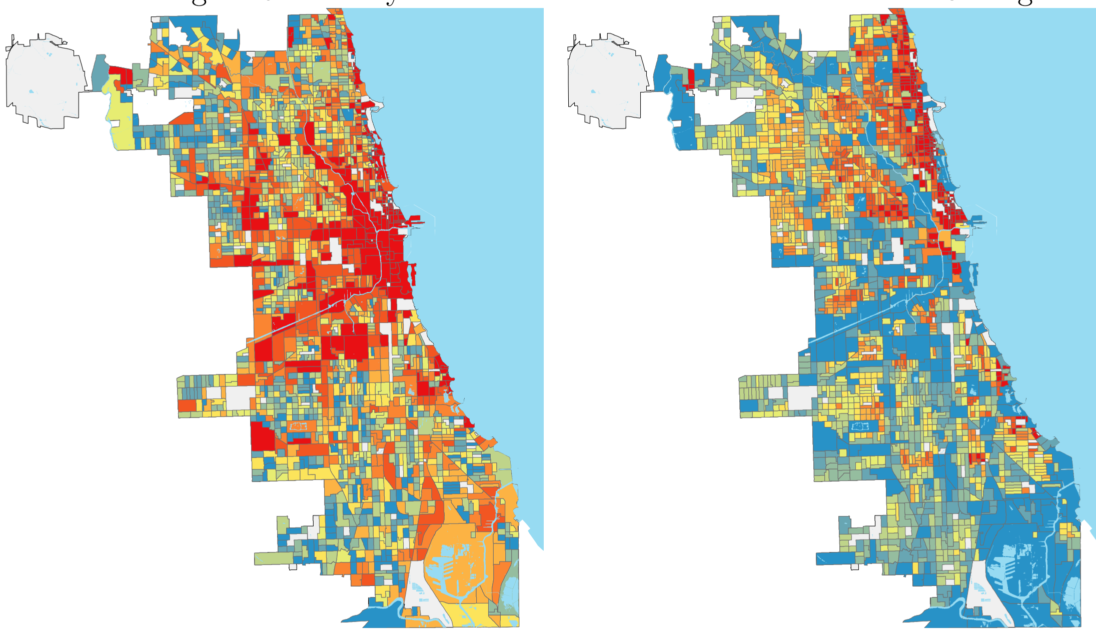
Notes: This figure shows the transportation infrastructure in Chicago that we use to calculate bilateral travel times. The infrastructure includes the "Metra" commuter rail (in blue, with stops indicated by blue circles), the "El" subway (in green, with stops indicated by green circles), the public bus routes (in purple, with stops indicated by purple circles), the interstate highways (in yellow), the expressways (in red), and the surface streets (in blue).

Figure 2: Zoning and buildings in Chicago



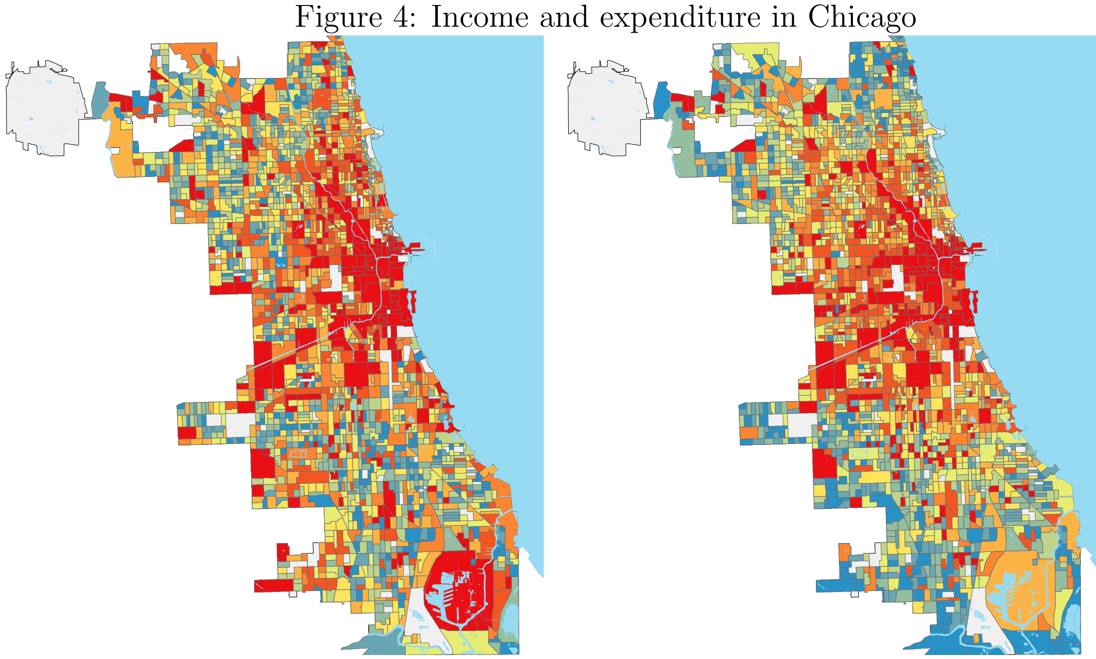
Notes: The left panel depicts the location, shape, and number of stories of all 820,944 buildings in Chicago (taller buildings are indicated in red and purple, shorter buildings in yellow). We recommend you zoom in to see the true level of detail. The right panel depicts the current zoning in Chicago. Together, we can construct the total square footage of residential and commercial space in Chicago.

Figure 3: Density of commercial and residential areas in Chicago



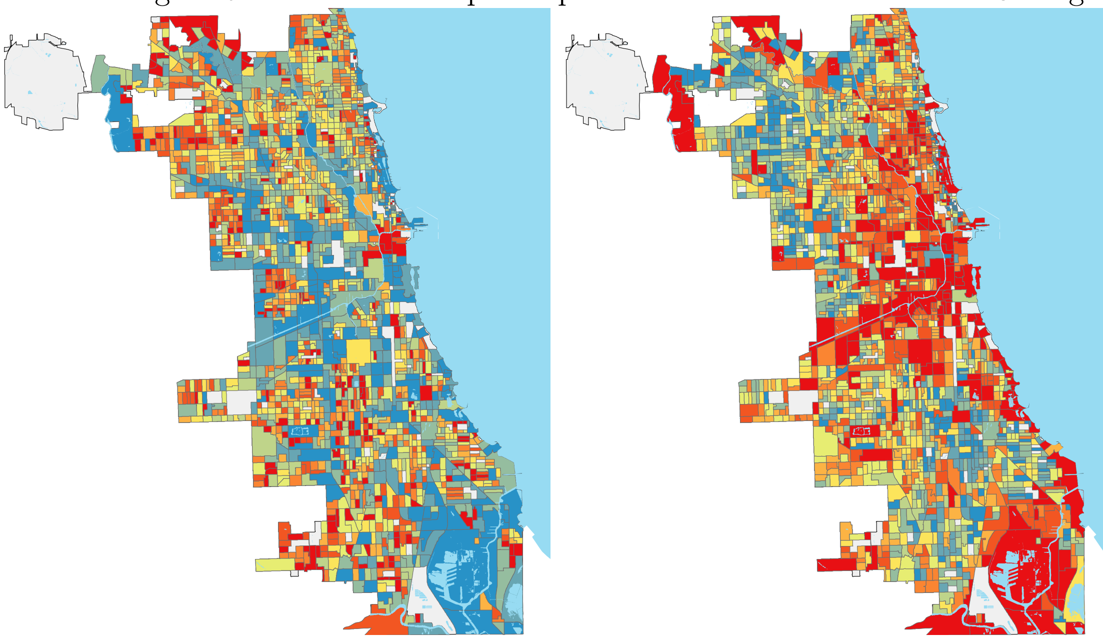
Notes: The left panel depicts the observed floor space devoted to commercial uses (H_i^F) and the right panel depicts the observed floor space (H_i^R) devoted to residential uses. The colors indicate the decile of the Census block group (normalized by area of the block group), with red (blue) indicated a higher (lower) decile.

Figure 4: Income and expenditure in Chicago



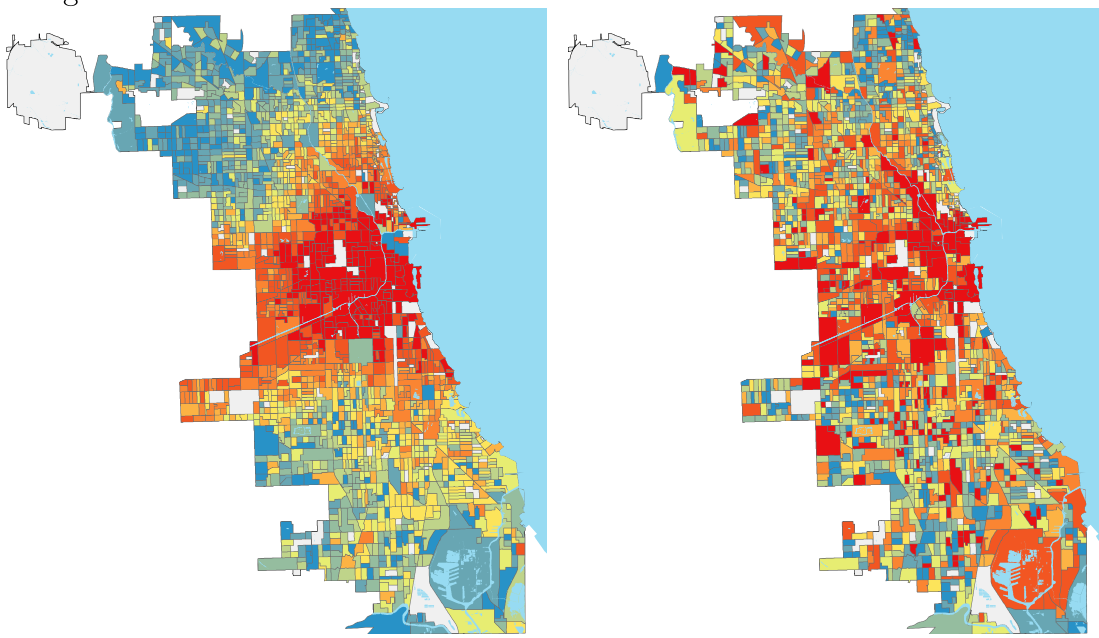
Notes: The left panel depicts the distribution of income of firms across Chicago (Y_i) and the right panel depicts the distribution of expenditure of residents across Chicago (E_i). The colors indicate the decile of the Census block group, with red (blue) indicated a higher (lower) decile.

Figure 5: Estimated composite productivities and amenities in Chicago



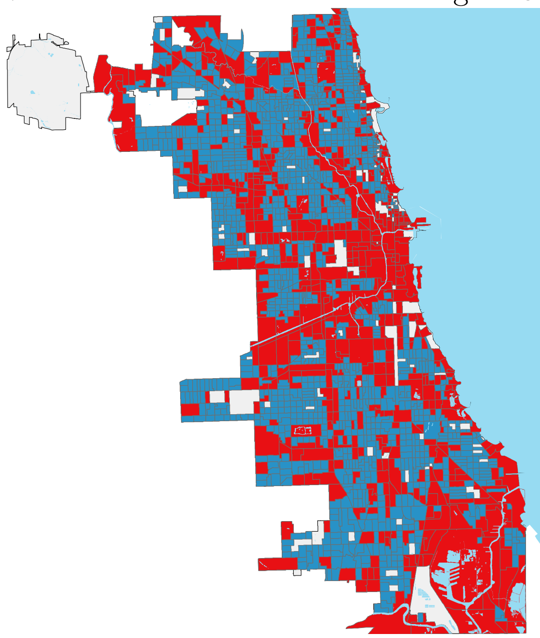
Notes: The left panel depicts the estimated composite productivity of each Census block group and the right panel depicts the estimated composite amenity of each Census block group. The colors indicate the decile of the Census block group, with red (blue) indicated a higher (lower) decile.

Figure 6: Estimated elasticity of welfare to increasing commercial and residential space in Chicago



Notes: The left panel depicts the estimated elasticity of city-wide welfare to an increase in the commercial space available in each Census block group. The right panel depicts the estimated elasticity of city-wide welfare to an increase in the residential space available in each Census block group. The colors in both figures indicate the decile of the elasticity, with red indicating a greater elasticity and blue indicating a lower elasticity.

Figure 7: The welfare effects of re-zoning in Chicago



Notes: This figure indicates the direction in which each Census block groups in Chicago ought to be rezoned in order to increase city-wide welfare. Red block groups are those for which more space should be reallocated to residential purposes and blue block groups are those for which more space should be reallocated for commercial purposes.

Appendix to "Optimal City Structure"

Treb Allen, Costas Arkolakis, Xiangliang Li

A Theory Appendix

A.1 Proof of Theorem 1

Proof. We design an equivalent setting to apply the first fundamental theorem of welfare economics. In the new setting, we modify the agent's endowments, behaviors and utility, but the agents will make exactly the same decision as the setting of the main context.

Agent's endowment of the firms and buildings is the same with the setting of the main context. However, their time endowment is different. They have time endowment $e_{ij}(\omega) = e(\omega) - t_{ij}$ for each commuting pair (i,j), which can be used to increase productivity, work and enjoy leisure, correspondingly the time used are denoted as $e_{ij}^A(\omega)$, $e_{ij}^w(\omega)$ and $e_{ij}^l(\omega)$. According to condition (ii) of the theorem, we can define the largest effective labor $e_{ij}^e(\omega,t) = \max_{e_{ij}^A+e_{ij}^w\le t} A_\omega\left(e_{ij}^A(\omega),i,j\right)e_{ij}^w$ as a function of the total time t used in productivity improving and work; the effective labor endowment $\tilde{e}_{ij}(\omega) = \max_{t\le e_{ij}(\omega)} e_{ij}^e(\omega,t)$. Furthermore, we denote the modified effective leisure as e_{ij}^l, and the corresponding real leisure as $l^{real}\left(\omega,e_{ij}^l\right) = \max\left\{t|\tilde{e}_{ij}(\omega) - e_{ij}^l - e_{ij}^e(\omega,e_{ij}(\omega)-t)\ge 0\right\}$

Besides, their utility function is different. Denote $\left\{\{g_{\theta k}(\omega)\}_{\theta\in\Theta,k\in S},\left\{h_k^R(\omega)\right\}_{k\in S},\left\{e_{ij}^l\right\}_{i,j\in S}\right\}$ as consumption and leisure plan $z(\omega)$, where $\{g_{\theta k}(\omega)\}_{\theta\in\Theta}$ is the goods consumed in k. And define $u_\omega\left(e_{ij}^l(\omega),\{g_{\theta i}(\omega)\}_{\theta\in\Theta},h_i^R(\omega),i,j\right) = -\infty$ if $h_i^R(\omega)=0$ or for all $\theta\in\Theta$ $g_{\theta i}(\omega)=0$. For any $z(\omega)$, his/her utility is

$$U(\omega) = \prod_{(k,l)\neq(I_{max}(z(\omega)),J_{max}(z(\omega)))} 1_{l^{real}\left(e_{mn}^l\right)\ge e_{mn}(\omega)} \max_{i,j} u_\omega\left(l^{real}\left(\omega,e_{ij}^l\right),\{g_{\theta i}(\omega)\}_{\theta\in\Theta},h_i^R(\omega),i,j\right)$$

where $(I_{max}(z(\omega)),J_{max}(z(\omega))) = \operatorname*{argmax}_{i,j} u_\omega\left(l^{real}\left(e_{ij}^l\right),\{g_{\theta i}(\omega)\}_{\theta\in\Theta},h_i^R(\omega),i,j\right)$, $g_{\theta i}(\omega) = \sum_{\delta\in\Delta_\theta}\frac{q_{\delta i}(\omega)}{d_{k(\delta)i}}$ is the goods he get at location i in which $q_{\delta i}(\omega)$ satisfies the constraint $\sum_{i\in S} q_{\delta i}(\omega)\le q_\delta(\omega)$.

Our modified agent make the decision based on

$$\max_{\{q_\delta(\omega)\}_{\delta\in\Delta,i\in S},\left\{h_i^R(\omega)\right\}_{i\in S},e_{ij}^l} U(\omega)$$

$$\text{s.t.} \sum_{\delta\in\Delta} p_\delta q_\delta(\omega) + \sum_{i\in S} r_i^R h_i^R(\omega) + \sum_{m,n\in S} w_n e_{mn}^l \quad \le$$

$$\le \sum_{\delta\in\Delta} s_\delta(\omega)\pi_\delta + \sum_{i\in S} r_i^R \left[s_i^R(\omega) + s_i^F(\omega)\right] + \sum_{m,n\in S} w_n \tilde{e}_{mn}(\omega)$$

where $q_{\delta i}(\omega)$ is how much goods bought from firm δ are transported to location i. We can verify that the modified agents behaves exactly the same as the original agents.

Firms and the market clearing conditions are also the same with the setting of the main context. .

Define the allocation $\{\{x_\omega\},\{y_\delta\}\}$ of the system. Both x_ω and y_δ contain three cells: the first cell are the goods quantity and firm share; the second cell are the building; the third share is effective time(labor). Specifically, $x_\omega = \left\{\{(q_\delta(\omega),0)\}_{\delta\in\Delta}, \left\{h_i^R(\omega)\right\}_{i\in S}, \left\{e_{mn}^l(\omega)\right\}_{m,n\in S}\right\}$ and $y_\delta = \{\{(0,0),..,(0,0),(Y_\delta,-1),(0,0),..,(0,0)\},\{0,...,0,-h(\delta),0\},\{0,...,0\}\}$. Also the endowment of the society is $e = \left\{\{0,0\}_{\delta\in\Delta}, \{H_i\}_{i\in S}, \left\{\sum_{k\in S}\int_{\omega\in\Omega} e_{kl}^e(\omega)\,d\omega\right\}\right\}$,notice in condition (ii) the assumption of $t_{ij}(\omega)$ makes sure that the last cell of e is a constant. Finally, the corresponding price is $p = \left\{\{(p_\delta,\pi_\delta)\}_{\delta\in\Delta}, \left\{r_i^R\right\}_{i\in S}, \{w_k\}_{k\in S}\right\}$.

In condition (i), the assumption of local non-satiation of original utility preferences implies local non-satiation of modified utility. Thus first fundamental theorem of welfare economics applies.

□

A.2 Derivation of Agent's optimal choices

Step 1: Agent's consumption over goods and housing

Given working time e^w and productivity $A(\omega)$, the first, agent's sub-problem is

$$\max_{\{q_k\}_{k\in S},h_R} Q_i(\omega)^\beta \left(h_i^R(\omega)\right)^{1-\beta} = \left(\sum_k q_{ki}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\beta\sigma}{\sigma-1}} \left(h_i^R(\omega)\right)^{1-\beta}$$

subject to

$$\sum_k \tau_{ki} p_{ki}(\omega) q_{ki}(\omega) + r_i^R h_i^R(\omega) \le \frac{1}{\alpha\beta} w_j A(\omega) e^w,$$

where the ratio $1/\alpha\beta$ arises because of capital income. After some algebra, we get obtain

$$\max Q_i^{\beta}\left(h_i^R\right)^{1-\beta}=\frac{w_j e^w A(\omega)}{\alpha\beta P_i^{\beta}\left(r_i^R\right)^{1-\beta}} \tag{51}$$

where P_i is the corresponding CES price index which is defined in (29).

Step 2: Agent allocation over interaction with others

Given time spend in interactions e^A, the agent's sub-problem is

$$\max_{\left\{l_k^A(\omega)\right\}_{k\in S}} A(\omega)=\bar{A}_j\left[\sum_k\left(\left(L_k^E\right)^{\eta} l_k^A(\omega)^{\mu}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

subject to

$$\sum_k t_{jk} l_k^A(\omega)\leq e^A,$$

which is equivalent with maximizing

$$\max_{\left\{l_k^A(\omega)\right\}_{k\in S}}\left[\sum_k\left(\left(L_k^E\right)^{\frac{\eta}{\mu}} l_k^A(\omega)\right)^{\frac{\mu(\varepsilon-1)}{\varepsilon}}\right]^{\frac{\varepsilon}{\mu(\varepsilon-1)}},$$

subject to the constraint.

Setting up the Lagrange function, we obtain the first order condition

$$\left[\sum_k\left(\left(L_k^E\right)^{\frac{\eta}{\mu}} l_k^A(\omega)\right)^{\frac{\mu(\varepsilon-1)}{\varepsilon}}\right]^{\frac{\varepsilon}{\mu(\varepsilon-1)}-1}\left(L_k^E\right)^{\frac{\eta(\varepsilon-1)}{\varepsilon}}\left[l_k^A(\omega)\right]^{\frac{\mu(\varepsilon-1)}{\varepsilon}-1}=\lambda t_{jk}.$$

After some algebra, we get

$$A(\omega)=A_j\left(e^A\right)^{\mu}, \tag{52}$$

where we define

$$A_j=\bar{A}_j\left[\sum_k\left(\left(L_k^E\right)^{\eta} t_{jh}^{-\mu}\right)^{\frac{\epsilon-1}{\epsilon-\mu(\epsilon-1)}}\right]^{\frac{\epsilon-\mu(\epsilon-1)}{\varepsilon-1}}. \tag{53}$$

Step 3: Agent allocating time among e^l, e^A, e^w

Now we substitute the solutions for the expressions of $Q_i(\omega)^{\beta}\left(h_i^R(\omega)\right)^{1-\beta}$ and $A(\omega)$, equations (51) and (53), in the utility function $u(\omega)=\bar{u}_i\upsilon_{ij}(\omega)\, l^{\beta_l} Q_i(\omega)^{\beta}\left(h_i^R(\omega)\right)^{1-\beta}$, the problem becomes

$$\max_{e^l,e^A,e^w} \frac{\bar{u}_i \upsilon_{ij}(\omega) w_j A_j}{\alpha\beta P_i^\beta (r_i^R)^{1-\beta}} \left(e^l\right)^{\beta_l} e^w \left(e^A\right)^\mu$$

subject to (taking the commuting choice, t_{ij}, as given) equation (5). From this optimization we obtain fraction of time spend in leisure $e_{ij}^l = \frac{\beta_l(1-t_{ij})}{1+\beta_l+\mu}$, fraction of time spend in working, $e_{ij}^w = \frac{1-t_{ij}}{1+\beta_l+\mu}$, and fraction of time spending in interacting, $e_{ij}^A = \frac{\mu(1-t_{ij})}{1+\beta_l+\mu}$, and agent's productivity $A_{ij} = \mu^\mu A_j \left(\frac{e-t_{ij}}{1+\beta_l+\mu}\right)^\mu$.

Substituting these solutions to the optimization above we have,

$$u(\omega) = c\bar{u}_i \upsilon_{ij}(\omega) \frac{w_j A_j}{d_{ij} P_i^\beta (r_i^R)^{1-\beta}} \tag{54}$$

where $d_{ij} = \left(\frac{1-t_{ij}}{1+\beta_l+\mu}\right)^{-(1+\mu+\beta_l)}$ and c is a constant which does not matter for our solution and we ignore it henceforth. Using this last expression, and aggregating across agents with the Frechet distribution we obtain expression (24) in the main text, which completes the derivations.

A.3 Proof of Corollary 1

Proof. Suppose the solution of $\mathcal{F}_2$ is $\left\{r_i^R, p_i, P_i, L_i^E, A_i H_j^F, H_i^R, U\right\}$. Then we know all the decisions of agent ω: commuting choice$(I(\omega), J(\omega))$, productivity $A(\omega)$, working time $e^w(\omega)$. Thus, agent ω's total income is $E(\omega) = \frac{1}{\alpha\beta} w_{J(\omega)} A(\omega) e^w(\omega)$. Now we design an initial capital allocation plan: allocate the amount of $h(\omega) = \frac{(1-\beta)E(\omega)}{r_{I(\omega)}^R}$ building in location $I(\omega)$ to agent ω (exactly the total footage agent lives in) and also allocate the amount of $\frac{\beta(1-\alpha)E(\omega)}{r_{J(\omega)}^F}$ building in location $J(\omega)$ to agent ω . It is easy to be verify that all the buildings are exactly allocated to all the agents. If $\eta = 0$, under this capital allocation plan the parametric model is a special case of the above general model. Denote the equilibrium under this setting is $\tilde{\mathcal{F}}_2$. Obviously, $\left\{r_i^R, p_i, P_i, L_i^E, A_i H_j^F, H_i^R, U\right\}$ is also the solution of $\tilde{\mathcal{F}}_2$. From Theorem 1, we know that this solution is efficient. □

A.4 Proof of Theorem 2

Before the formal proof, we transform the equilibrium a little bit and make some notations. We write equilibrium $\mathcal{F}_1$ (equations (28)-(11)) as the following set of equations.

$$\left(r_i^R\right)^{(1-\beta)\theta+1} P_i^{\beta\theta} = \lambda \sum_j K_{ij}^{r_R} p_j^{\theta+1} \left(L_j^E\right)^{(\alpha-1)(\theta+1)} A_j^{\theta+1} \tag{55}$$

$$\left(L_i^E\right)^{1+\theta(1-\alpha)} p_i^{-\theta} A_i^{-1-\theta} = \lambda \sum_j K_{ij}^{L_E} P_j^{-\beta\theta} \left(r_j^R\right)^{-(1-\beta)\theta}, \tag{56}$$

$$p_i^\sigma \left(L_i^E\right)^\alpha = \sum_j K_{ij}^p r_j^R P_j^{\sigma-1}, \tag{57}$$

$$P_i^{1-\sigma} = \sum_j K_{ij}^P p_j^{1-\sigma}, \tag{58}$$

$$A_i^{\tilde{\epsilon}} = \sum_j K_{ij}^A \left(L_j^E\right)^{\eta\tilde{\epsilon}} \tag{59}$$

and

$$1 = \sum_{i,j} \lambda \left(\frac{\bar{u}_i A_j \left(\alpha p_j L_{Ej}^{\alpha-1} H_{Fj}^{1-\alpha}\right)}{d_{ij} P_i^\beta \left(r_i^R\right)^{1-\beta}} \right)^\theta \tag{60}$$

where $\tilde{\varepsilon} = \frac{\epsilon-1}{\epsilon-\mu(\varepsilon-1)}$, $\lambda = \frac{\bar{L}}{U^\theta}$, $K_{ij}^{r_R} = \frac{1-\beta}{\alpha\beta}\alpha^{1+\theta} e_{ij}^w e_{ij}^A \left(H_i^R\right)^{-1} \bar{u}_i^\theta \left(H_j^F\right)^{(1-\alpha)(1+\theta)} d_{ij}^{-\theta}$, $K_{ij}^{L_E} = \alpha^\theta e_{ji}^w e_{ji}^A \bar{u}_j^\theta \left(H_i^F\right)^{(1-\alpha)\theta} d_{ji}^{-\theta}$, $K_{ij}^p = \frac{\beta}{1-\beta}\left(H_i^F\right)^{\alpha-1} H_j^R \tau_{ij}^{1-\sigma}$, $K_{ij}^P = \tau_{ji}^{1-\sigma}$ and $K_{ij}^A = \bar{A}_i t_{ij}^{-\mu\tilde{\epsilon}}$. Notice that for the convenience of proving, we use p_i instead of w_i unlike the context, and they can easily be transformed from one to the other by $w_i = \alpha p_i \left(L_i^E\right)^{\alpha-1}\left(H_i^F\right)^{1-\alpha}$.

We define the corresponding coefficient matrix

$$\mathbf{\Gamma} = \begin{pmatrix} (1-\beta)\theta+1 & 0 & 0 & \beta\theta & 0 \\ 0 & 1+\theta(1-\alpha) & -\theta & 0 & -1-\theta \\ 0 & \alpha & \sigma & 0 & 0 \\ 0 & 0 & 0 & 1-\sigma & 0 \\ 0 & 0 & 0 & 0 & \tilde{\epsilon} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & (\alpha-1)(\theta+1) & \theta+1 & 0 & \theta+1 \\ (\beta-1)\theta & 0 & 0 & -\beta\theta & 0 \\ 1 & 0 & 0 & \sigma-1 & 0 \\ 0 & 0 & 1-\sigma & 0 & 0 \\ 0 & \eta\tilde{\epsilon} & 0 & 0 & 0 \end{pmatrix},$$

Proof. Now we mainly use theorem 1 of Allen, Arkolakis, and Li (2015) to prove this theorem.

Part i):

The existence proof proceeds in two steps. First, as $\tilde{\epsilon}$ is non-zero, it is obvious that $\mathbf{\Gamma}$ is invertible, then according to Theorem 1 of Allen, Arkolakis, and Li (2015) there exists a solution for the following system.

$$\left(r_i^R\right)^{(1-\beta)\theta+1} P_i^{\beta\theta} = \lambda_1 \sum_j K_{ij}^{r_R} p_j^{\theta+1} \left(L_j^E\right)^{(\alpha-1)(\theta+1)} A_j^{\theta+1}$$

$$\left(L_i^E\right)^{1+\theta(1-\alpha)} p_i^{-\theta} A_i^{-1-\theta} = \lambda_2 \sum_j K_{ij}^{L_E} P_j^{-\beta\theta} \left(r_j^R\right)^{-(1-\beta)\theta},$$

$$p_i^{\sigma} \left(L_i^E\right)^{\alpha} = \lambda_3 \sum_j K_{ij}^{p} r_j^R P_j^{\sigma-1},$$

$$P_i^{1-\sigma} = \lambda_4 \sum_j K_{ij}^{P} p_j^{1-\sigma},$$

$$A_i^{\tilde{\epsilon}} = \lambda_5 \sum_j K_{ij}^{A} \left(L_j^E\right)^{\eta\tilde{\epsilon}}$$

where $\lambda_k (k = 1, ..., 5)$ can be any constants (eigenvalues) as long as they together with r_i^R, p_i, P_i, L_i^E, A_i can make the above four equations hold.

Second, we show that we can obtain the solutions of equations 55 to 60 from the solution of above equations by scaling up r_i^R, p_i, P_i, L_i^E, A_i. Thus, the existence of $\mathcal{F}_1$ is established. The details of transformation and scaling are as follows.

We make the following transformations $\begin{pmatrix} \ln x_{1,i} \\ \ln x_{2,i} \\ \ln x_{3,i} \\ \ln x_{4,i} \\ \ln x_{5,i} \end{pmatrix} = \mathbf{\Gamma} \begin{pmatrix} \ln r_i^R \\ \ln L_i^E \\ \ln p_i \\ \ln P_i \\ \ln A_i \end{pmatrix}$ i.e. $x_{1,i} = \left(r_i^R\right)^{(1-\beta)\theta+1} P_i^{\beta\theta}$, $x_{2,i} = \left(L_i^E\right)^{1+\theta(1-\alpha)} p_i^{-\theta} A_i^{-1-\theta}$, $x_{3,i} = p_i^{\sigma} \left(L_i^E\right)^{\alpha}$, $x_{4,i} = P_i^{1-\sigma}$ and $x_{5,i} = A_i^{\tilde{\epsilon}}$. Then we have

$$\begin{pmatrix} p_j^{\theta+1} \left(L_j^E\right)^{(\alpha-1)(\theta+1)} A_j^{\theta+1} \\ P_j^{-\beta\theta} \left(r_j^R\right)^{-(1-\beta)\theta} \\ r_j^R P_j^{\sigma-1} \\ p_j^{1-\sigma} \\ \left(L_i^E\right)^{\eta\tilde{\epsilon}} \end{pmatrix} = \exp \left[\mathbf{B\Gamma^{-1}} \begin{pmatrix} \ln x_{1,j} \\ \ln x_{2,j} \\ \ln x_{3,j} \\ \ln x_{4,j} \\ \ln x_{5,j} \end{pmatrix} \right].$$

Particularly, we have a solution of $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ that is a fixed point of operator $T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix}$ (fixed point is guaranteed according to Theorem 1 of Allen, Arkolakis, and Li (2015)) which is defined as follows.: $T_k : R_{++}^{5N} \to R_{++}^{N}$ $k = 1, 2, 3, 4, 5$ such that $(T_1(x))_i =$

$\frac{\sum_j K_{ij}^{r_R} x_{1,j}^{m_{11}} x_{2,j}^{m_{12}} x_{3,j}^{m_{13}} x_{4,j}^{m_{14}} x_{5,j}^{m_{15}}}{\sum_i \sum_j K_{ij}^{r_R} x_{1,j}^{m_{11}} x_{2,j}^{m_{12}} x_{3,j}^{m_{13}} x_{4,j}^{m_{14}} x_{5,j}^{m_{15}}}$ and $(T_2(x))_i = \frac{\sum_j K_{ij}^{L_E} x_{1,j}^{m_{21}} x_{2,j}^{m_{22}} x_{3,j}^{m_{23}} x_{4,j}^{m_{24}} x_{5,j}^{m_{25}}}{\sum_i \sum_j K_{ij}^{L_E} x_{1,j}^{m_{21}} x_{2,j}^{m_{22}} x_{3,j}^{m_{23}} x_{4,j}^{m_{24}} x_{5,j}^{m_{25}}}$ and similarly for $T_k(x)$ $k = 3, 4, 5$ where $m_{ij} = (\mathbf{B\Gamma^{-1}})_{ij}$.

Denote $a_1 = \sum_i \sum_j K_{ij}^{r_R} x_{1,j}^{m_{11}} x_{2,j}^{m_{12}} x_{3,j}^{m_{13}} x_{4,j}^{m_{14}} x_{5,j}^{m_{15}}$, $a_2 = \sum_i \sum_j K_{ij}^{L_E} x_{1,j}^{m_{21}} x_{2,j}^{m_{22}} x_{3,j}^{m_{23}} x_{4,j}^{m_{24}} x_{5,j}^{m_{25}}$ and similarly for a_k $k = 3, 4, 5$

Denote $\begin{pmatrix} r^{R0} \\ L^{E0} \\ p^0 \\ P^0 \\ A^0 \end{pmatrix} = \exp\left[\mathbf{\Gamma}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}\right]$. In the following, we will show that there exists "scale" variables $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}$ (b_i is a constant) and $\begin{pmatrix} r^R \\ L^E \\ p \\ P \\ A \end{pmatrix}$ that satisfies $\begin{pmatrix} r^{R0} \\ L^{E0} \\ p^0 \\ P^0 \\ A^0 \end{pmatrix} = \begin{pmatrix} b_1 r^R \\ b_2 L^E \\ b_3 p \\ b_4 P \\ b_5 A \end{pmatrix}$ such that $\begin{pmatrix} r^R \\ L^E \\ p \\ P \\ A \end{pmatrix}$ is the solution of $\mathcal{F}_1$. It is equivalent by showing the the following "level" equations has a solution of $\ln b$ and $\ln \lambda$.

$$\begin{pmatrix} (1-\beta)\theta + 1 & -(\alpha-1)(\theta+1) & -\theta-1 & \beta\theta & -\theta-1 \\ (1-\beta)\theta & 1+\theta(1-\alpha) & -\theta & \beta\theta & -1-\theta \\ -1 & \alpha & \sigma & 1-\sigma & 0 \\ 0 & 0 & \sigma-1 & 1-\sigma & 0 \\ 0 & -\eta\tilde{\epsilon} & 0 & 0 & \tilde{\epsilon} \\ \theta(1-\beta) & \theta(1-\alpha) & -\theta & \theta\beta & -\theta \end{pmatrix} \ln \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \ln \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} + \ln \begin{pmatrix} \lambda \\ \lambda \\ 1 \\ 1 \\ 1 \\ \lambda \end{pmatrix} \tag{61}$$

where $a_6 = 1/\sum\sum (1+\beta_l+\mu)^{\theta(1+\mu+\beta_l)} (e-t_{ij})^{\theta(1+\mu+\beta_l)} \left(\frac{\bar{u}_i A_j w_j}{P_i^\beta \left(r_i^R\right)^{1-\beta}}\right)^\theta$.

It is a linear equation, but the corresponding matrix of $\ln b$ and $\ln \lambda$ happen to singular. So we have to show it has a solution by solving. Actually, the following expression solves the above equation(to keep the proof of Theorem 2 concise, we leave the solving in the end of this section)

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \lambda \end{pmatrix} = \begin{pmatrix} \frac{a_1}{a_2}\left(\frac{a_2}{a_6}a_5^{\frac{1}{\tilde{\epsilon}}}\right)^{\frac{\alpha}{1-\eta}} \\ \left(\frac{a_2}{a_6}a_5^{\frac{1}{\tilde{\epsilon}}}\right)^{\frac{1}{1-\eta}} \\ 1 \\ a_4^{\frac{1}{1-\sigma}} \\ \frac{a_6}{a_2}\left(\frac{a_2}{a_6}a_5^{\frac{1}{\tilde{\epsilon}}}\right)^{\frac{1}{1-\eta}} \\ b_1^{(1-\beta)\theta} b_2^{\theta(1-\alpha)} b_4^{\beta\theta} b_5^{-\theta} a_6^{-1} \end{pmatrix}$$

Part ii): In the following, we are going to verify the spectra radius of $|\mathbf{B\Gamma^{-1}}|$ to establish the uniqueness.

It is obvious that $\mathbf{\Gamma} = P_{\tilde{\epsilon}}\tilde{\mathbf{\Gamma}}$ and $\mathbf{B} = P_{\tilde{\epsilon}}\tilde{\mathbf{B}}$ where

$$\tilde{\mathbf{\Gamma}} = \begin{pmatrix} (1-\beta)\theta+1 & 0 & 0 & \beta\theta & 0 \\ 0 & 1+\theta(1-\alpha) & -\theta & 0 & -1-\theta \\ 0 & \alpha & \sigma & 0 & 0 \\ 0 & 0 & 0 & 1-\sigma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{\mathbf{B}} = \begin{pmatrix} 0 & (\alpha-1)(\theta+1) & \theta+1 & 0 & \theta+1 \\ (\beta-1)\theta & 0 & 0 & -\beta\theta & 0 \\ 1 & 0 & 0 & \sigma-1 & 0 \\ 0 & 0 & 1-\sigma & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \end{pmatrix},$$

and

$$P_{\tilde{\epsilon}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\epsilon} \end{pmatrix}.$$

Notice that $\rho\left(|\mathbf{B\Gamma^{-1}}|\right) = \rho\left(\left|P_{\tilde{\epsilon}}\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}P_{\tilde{\epsilon}}^{-1}\right|\right) = \rho\left(P_{|\tilde{\epsilon}|}\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|P_{|\tilde{\epsilon}|}^{-1}\right) = \rho\left(\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|\right)$ where $P_{|\tilde{\epsilon}|} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & |\tilde{\epsilon}| \end{pmatrix}$. Also $P_{|\tilde{\epsilon}|}^{-1} = \left|P_{\tilde{\epsilon}}^{-1}\right|$.

After some calculation by hand and computer, we get

$$\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right| = \begin{pmatrix} 0 & \frac{(\alpha+\sigma-\alpha\sigma)(\theta+1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} & \frac{\theta+1}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} & 0 & \frac{\alpha(\sigma-1)(\theta+1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} \\ \frac{(1-\beta)\theta}{(1-\beta)\theta+1} & 0 & 0 & \frac{\beta\theta}{[(1-\beta)\theta+1](\sigma-1)} & 0 \\ \frac{1}{(1-\beta)\theta+1} & 0 & 0 & \frac{|\beta\theta-((1-\beta)\theta+1)(\sigma-1)|}{[(1-\beta)\theta+1](\sigma-1)} & 0 \\ 0 & \frac{\alpha(\sigma-1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} & \frac{(\sigma-1)((1-\alpha)\theta+1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} & 0 & \frac{\alpha(\sigma-1)(\theta+1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} \\ 0 & \frac{|\eta|\sigma}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} & \frac{|\eta|\theta}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} & 0 & \frac{|\eta|\sigma(\theta+1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} \end{pmatrix}.$$

If $\beta \leq \frac{\theta+1}{\theta}\frac{\sigma-1}{\sigma}$, obviously $\begin{pmatrix} 1 & 1 & 1 & 1 & \frac{2\alpha(\sigma-1)(\theta+1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} \end{pmatrix}\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right| = \begin{pmatrix} 1 & 1 & 1 & 1 & \frac{2\alpha(\sigma-1)(\theta+1)}{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta} \end{pmatrix}$ which means 1 is its largest eigenvalue. Proof is done for this part.

In fact, we also know about the spectra radius when $|\eta| \neq 0$. From the Perron-Frobenius theorem, we know that there exists a left positive eigenvector v_L and v_R corresponding the largest eigenvalue $\rho\left(\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|\right)$. Besides, from corollary 2.4 on page 185 of Stewart and Sun (1990), we have $\frac{\partial\rho\left(\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|\right)}{\partial m_{ij}} = \frac{v_{Li}v_{Rj}}{v_L^T v_R}$ where m_{ij} is the (i,j)th element of $\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|$. $\rho\left(\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|\right)$ is always increasing with respect to $|\eta|$. Furthermore, notice that when $|\eta| > 0$, $\tilde{M}_K$ is non-negative irreducible, again according to Perron-Frobenius theorem, v_L and v_R are positive. Thus $\rho\left(\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|\right)$ is increasing with respect to $|\eta|$. So for $|\eta| \neq 0$, we always have $\rho\left(\left|\tilde{\mathbf{B}}\tilde{\mathbf{\Gamma}}^{-1}\right|\right) > 1$, we are not sure about the uniqueness only from the parameter coefficient. However, we impose some restrictions on kernels we can get the uniqueness for $|\eta| \neq 0$, an example is part iii.

Part iii: Similarly with part ii, in the following, we also prove the uniquness by verifying the spectra radius of corresponding coefficient matrix.

If $\tau_{ij} = 1$, normalize $P_i \equiv 1$ for all i(we have $\sum_{k=1}^{N} p_k^{1-\sigma} = 1$). The equation (28) can be transformed into

$$Y_i p_i^{\sigma-1} = \beta E \Rightarrow$$

$$p_i^{\sigma} L_{Ei}^{\alpha} H_{Fi}^{1-\alpha} = \beta E$$

where $E = \sum_j E_j$.

Thus $p_i = \left(L_{Ei}^{-\alpha} H_{Fi}^{\alpha-1}\beta E\right)^{\frac{1}{\sigma}} = \beta^{\frac{1}{\sigma}} E^{\frac{1}{\sigma}} L_{Ei}^{-\frac{\alpha}{\sigma}} H_{Fi}^{\frac{\alpha-1}{\sigma}}$. Equilibrium conditions can be written as

$$\left(r_i^R\right)^{(1-\beta)\theta+1} = \lambda\beta^{\frac{\theta+1}{\sigma}} E^{\frac{\theta+1}{\sigma}} \sum_j K_{ij}^{r_R} H_{Fj}^{\frac{(\alpha-1)(\theta+1)}{\sigma}} \left(L_j^E\right)^{-\frac{\sigma-\sigma\alpha+\alpha}{\sigma}(\theta+1)} A_j^{\theta+1} \tag{62}$$

$$\left(L_i^E\right)^{1+\theta\left(\frac{\sigma-\sigma\alpha+\alpha}{\sigma}\right)} A_i^{-1-\theta} = \lambda\beta^{\frac{\theta}{\sigma}} E^{\frac{\theta}{\sigma}} \sum_j K_{ij}^{L_E} H_{Fi}^{\frac{\theta(\alpha-1)}{\sigma}} \left(r_j^R\right)^{-(1-\beta)\theta}, \tag{63}$$

$$A_i^{\tilde{\epsilon}} = \sum_j K_{ij}^A \left(L_j^E\right)^{\eta\tilde{\epsilon}} \tag{64}$$

combined with two normalized conditions: total labor constraint and price condition $\sum_{k=1}^N p_k^{1-\sigma} = 1$.

Thus the corresponding coefficient matrix are (without causing ambiguity, we are using the same matrix notation with above.)

$$\mathbf{\Gamma} = \begin{pmatrix} (1-\beta)\theta + 1 & 0 & 0 \\ 0 & 1+\theta\left(\frac{\sigma-\sigma\alpha+\alpha}{\sigma}\right) & -1-\theta \\ 0 & 0 & \tilde{\epsilon} \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 & -\frac{\sigma-\sigma\alpha+\alpha}{\sigma}(\theta+1) & \theta+1 \\ (\beta-1)\theta & 0 & 0 \\ 0 & \eta\tilde{\epsilon} & 0 \end{pmatrix},$$

Similarly with part ii, the eigenvalues of $|\mathbf{B\Gamma^{-1}}|$ are the same with ($\tilde{\epsilon}$ does not matter here)

$$\mathbf{M} = \begin{pmatrix} 0 & \frac{(\sigma-\sigma\alpha+\alpha)(\theta+1)}{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta} & \frac{\alpha(\sigma-1)(\theta+1)}{(\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta)} \\ \left|\frac{\theta(\beta-1)}{\theta-\beta\theta+1}\right| & 0 & 0 \\ 0 & \frac{|\eta|\sigma}{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta} & \frac{|\eta|\sigma(\theta+1)}{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta} \end{pmatrix}$$

As long as $|\eta| > 0$, $\mathbf{M}$ is irreducible. According to Facts 7 of Section 9.2 in Hogben (2006), as long as there exists some $x \geq 0$ such that Here we user a test vector $\begin{pmatrix} 1 & 1 & x \end{pmatrix}\mathbf{M} \leq \begin{pmatrix} 1 & 1 & x \end{pmatrix}$. The spectra radius of $\mathbf{M}$ is no bigger than 1. A sufficient condition to make the spectral radius be no bigger than 1 is $\begin{pmatrix} 1 & 1 & x \end{pmatrix}\mathbf{M} \leq \begin{pmatrix} 1 & 1 & x \end{pmatrix}$. Thus we get inequalities $x \leq \frac{\alpha(\sigma-1)}{|\eta|\sigma}$ and $0 \leq \frac{\alpha(\sigma-1)(\theta+1)}{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta-|\eta|\sigma(\theta+1)} \leq x$. To make there exists some $x \geq 0$ we only need $0 \leq \frac{\alpha(\sigma-1)(\theta+1)}{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta-|\eta|\sigma(\theta+1)} \leq \frac{\alpha(\sigma-1)}{|\eta|\sigma}$. Thus, $|\eta| \leq \frac{\sigma+\alpha\theta+\sigma\theta-\alpha\sigma\theta}{2\sigma(\theta+1)}$ guarantees the spectra radius of $\mathbf{M}$ no bigger than 1, thus from Theorem 1 of Allen, Arkolakis, and Li (2015)) holds.

□

Solving level equation (61)

$$\begin{pmatrix} (1-\beta)\theta+1 & -(\alpha-1)(\theta+1) & -\theta-1 & \beta\theta & -\theta-1 \\ (1-\beta)\theta & 1+\theta(1-\alpha) & -\theta & \beta\theta & -1-\theta \\ -1 & \alpha & \sigma & 1-\sigma & 0 \\ 0 & 0 & \sigma-1 & 1-\sigma & 0 \\ 0 & -\eta\tilde{\epsilon} & 0 & 0 & \tilde{\epsilon} \\ \theta(1-\beta) & \theta(1-\alpha) & -\theta & \theta\beta & -\theta \end{pmatrix} \ln \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \ln \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} + \ln \begin{pmatrix} \lambda \\ \lambda \\ 1 \\ 1 \\ 1 \\ \lambda \end{pmatrix}$$

The first row minus the second and the third row minus the fourth row we get

$$\begin{pmatrix} 1 & -\alpha & -1 & 0 & 0 \\ (1-\beta)\,\theta & 1+\theta\,(1-\alpha) & -\theta & \beta\theta & -1-\theta \\ -1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & \sigma-1 & 1-\sigma & 0 \\ 0 & -\eta\tilde{\epsilon} & 0 & 0 & \tilde{\epsilon} \\ \theta\,(1-\beta) & \theta\,(1-\alpha) & -\theta & \theta\beta & -\theta \end{pmatrix} \ln \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \ln \begin{pmatrix} a_1/a_2 \\ a_2 \\ a_3/a_4 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} + \ln \begin{pmatrix} 1 \\ \lambda \\ 1 \\ 1 \\ 1 \\ \lambda \end{pmatrix}$$

Add the first onto the third, we get

$$\begin{pmatrix} 1 & -\alpha & -1 & 0 & 0 \\ (1-\beta)\,\theta & 1+\theta\,(1-\alpha) & -\theta & \beta\theta & -1-\theta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma-1 & 1-\sigma & 0 \\ 0 & -\eta\tilde{\epsilon} & 0 & 0 & \tilde{\epsilon} \\ \theta\,(1-\beta) & \theta\,(1-\alpha) & -\theta & \theta\beta & -\theta \end{pmatrix} \ln \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \ln \begin{pmatrix} \frac{a_1}{a_2} \\ a_2 \\ \frac{a_1 a_3}{a_2 a_4} \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} + \ln \begin{pmatrix} 1 \\ \lambda \\ 1 \\ 1 \\ 1 \\ \lambda \end{pmatrix}$$

Minus the second row from the sixth row

$$\begin{pmatrix} 1 & -\alpha & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma-1 & 1-\sigma & 0 \\ 0 & -\eta\tilde{\epsilon} & 0 & 0 & \tilde{\epsilon} \\ \theta\,(1-\beta) & \theta\,(1-\alpha) & -\theta & \theta\beta & -\theta \end{pmatrix} \ln \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \ln \begin{pmatrix} \frac{a_1}{a_2} \\ \frac{a_2}{a_6} \\ \frac{a_1 a_3}{a_2 a_4} \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} + \ln \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \lambda \end{pmatrix}$$

One can verify that the following solves the 1-2, 4-6 equations,

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \lambda \end{pmatrix} = \begin{pmatrix} \frac{a_1}{a_2}\left(\frac{a_2}{a_6} a_5^{\frac{1}{\tilde{\epsilon}}}\right)^{\frac{\alpha}{1-\eta}} \\ \left(\frac{a_2}{a_6} a_5^{\frac{1}{\tilde{\epsilon}}}\right)^{\frac{1}{1-\eta}} \\ 1 \\ a_4^{\frac{1}{1-\sigma}} \\ \frac{a_6}{a_2}\left(\frac{a_2}{a_6} a_5^{\frac{1}{\tilde{\epsilon}}}\right)^{\frac{1}{1-\eta}} \\ b_1^{(1-\beta)\theta} b_2^{\theta(1-\alpha)} b_4^{\beta\theta} b_5^{-\theta} a_6^{-1} \end{pmatrix}$$

Also one can also show that $\frac{a_1 a_3}{a_2 a_4} = 1$ thus the 3rd equation also holds. Details of proving $\frac{a_1 a_3}{a_2 a_4} = 1$ is as follows

We will first solve $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} =$ and λ from the rest 5 non-redundant equations (we set $b_3 = 1$ due to the price normalization) and then show that $\frac{a_1 a_3}{a_2 a_4} = 1$.

Notice that by definition of $\begin{pmatrix} r^{R0} \\ L^{E0} \\ p^0 \\ P^0 \\ A^0 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$, we have the following equations

$$a_1 \left(r_i^{R0}\right)^{(1-\beta)\theta+1} \left(P_i^0\right)^{\beta\theta} = \sum_j K_{ij}^{r_R} \left(p_j^0\right)^{\theta+1} \left(L_j^{E0}\right)^{(\alpha-1)(\theta+1)} \left(A_j^0\right)^{\theta+1} \tag{65}$$

$$a_2 \left(L_i^{E0}\right)^{1+\theta(1-\alpha)} \left(p_i^0\right)^{-\theta} \left(A_i^0\right)^{-\theta-1} = \lambda \sum_j K_{ij}^{L_E} \left(P_j^0\right)^{-\beta\theta} \left(r_j^{R0}\right)^{-(1-\beta)\theta}, \tag{66}$$

$$a_3 \left(p_i^0\right)^{\sigma} \left(L_i^{E0}\right)^{\alpha} = \sum_j K_{ij}^{p} r_j^{R0} \left(P_j^0\right)^{\sigma-1}, \tag{67}$$

$$a_4 \left(P_i^0\right)^{1-\sigma} = \sum_j K_{ij}^{P} \left(p_j^0\right)^{1-\sigma}, \tag{68}$$

Multiply both sides of equation 67 by$\left(p_i^0\right)^{1-\sigma} H_{Fi}^{\alpha-1}$ and sum over all $i \in S$, we get

$$a_3 \sum_i Y_i^0 = \sum_j \frac{\beta H_{Rj} r_j^{R0}}{1-\beta} \sum_i \frac{\left(\tau_{ij} p_i^0\right)^{1-\sigma}}{P_j^{1-\sigma}} = a_4 \sum_j \frac{\beta H_{Rj} r_j^{R0}}{1-\beta}$$

where $Y_i^0 = p_i^0 \left(L_i^{E0}\right)^{\alpha} H_{Fi}^{\alpha-1}$ and the last equality comes from equation68.
Multiply both sides of equation 65 by$\left(r_i^{R0}\right)^{-(1-\beta)\theta+1} H_i^R$ and sum over all $i \in S$, we get

$$a_1 \sum_i \frac{H_i^R r_i^{R0}}{1-\beta} = \frac{1}{\alpha\beta} \sum_i \sum_j w_j^0 L_{ij}^{E0} = a_2 \frac{1}{\alpha\beta} \sum_j w_j^0 L_j^{E0}$$

where $w_i^0 = \alpha p_i^0 \left(L_i^{E0}\right)^{\alpha-1} \left(H_i^{F0}\right)^{1-\alpha}$, $L_{ij}^{E0} = e_{ij}^w \left(e_{ij}^A\right)^{\mu} A_j^0 \left(\frac{\bar{u}_i A_j^0 w_j^0}{d_{ij} \left(P_i^0\right)^{\beta} \left(r_i^{R0}\right)^{1-\beta}}\right)^{\theta}$. The last equality comes from multiplying both sides of equation 66 by $\left(L_i^{E0}\right)^{-\theta(1-\alpha)} \left(p_i^0\right)^{\theta} \left(A_i^0\right)^{\theta+1}$.

Sum up, $a_1 = a_2 \frac{\frac{1}{\alpha\beta}\sum_j w_j^0 L_j^{E0}}{\sum_i \frac{H_i^R r_i^{R0}}{1-\beta}}$, $a_3 = \frac{a_4 \sum_j \frac{\beta H_{Rj} r_j^{R0}}{1-\beta}}{\sum_i Y_i^0}$. Thus,

$$\frac{a_1 a_3}{a_2 a_4} = \frac{\frac{1}{\alpha\beta}\sum_j w_j^0 L_j^{E0}}{\sum_i Y_i^0} = 1.$$

A.5 Analytical results

Derivation of equation (35)

Without trade cost, the same with the above part (iii) proof of theorem (2), equilibrium conditions can be written as

$$\left(r_i^R\right)^{(1-\beta)\theta+1} = \lambda\beta^{\frac{\theta+1}{\sigma}} E^{\frac{\theta+1}{\sigma}} \sum_j K_{ij}^{r_R} H_{Fj}^{\frac{(\alpha-1)(\theta+1)}{\sigma}} \left(L_j^E\right)^{-\frac{\sigma-\sigma\alpha+\alpha}{\sigma}(\theta+1)} A_j^{\theta+1}$$

$$\left(L_i^E\right)^{1+\theta\left(\frac{\sigma-\sigma\alpha+\alpha}{\sigma}\right)} A_i^{-1-\theta} = \lambda\beta^{\frac{\theta}{\sigma}} E^{\frac{\theta}{\sigma}} \sum_j K_{ij}^{L_E} H_{Fi}^{\frac{\theta(\alpha-1)}{\sigma}} \left(r_j^R\right)^{-(1-\beta)\theta},$$

$$A_i^{\tilde{\epsilon}} = \sum_j K_{ij}^A \left(L_j^E\right)^{\eta\tilde{\epsilon}}$$

Furthermore, we assume $\beta = 1$, i.e. there is no residential market, the second equation can be simplified to

$$\left(L_i^E\right)^{1+\theta\left(\frac{\sigma-\sigma\alpha+\alpha}{\sigma}\right)} A_i^{-1-\theta} = \lambda E^{\frac{\theta}{\sigma}} \sum_j K_{ij}^{L_E} H_{Fi}^{\frac{\theta(\alpha-1)}{\sigma}}$$

Thus $A_i = \left(L_i^E\right)^{\frac{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta}{\sigma(1+\theta)}} \left(\lambda\beta^{\frac{\theta}{\sigma}} E^{\frac{\theta}{\sigma}} \sum_j K_{ij}^{L_E} H_{Fi}^{\frac{\theta(\alpha-1)}{\sigma}}\right)^{-\frac{1}{1+\theta}}$ it into the third equation, we get

$$l_i = \tilde{\lambda}\sum_j \tilde{K}_{ij}^A \left(l_j\right)^{\frac{\eta}{\sigma_1}}$$

where $l_i = \left(L_i^E\right)^{\sigma_1\tilde{\epsilon}}$, $\sigma_1 = \frac{-\alpha\theta\sigma+\theta\sigma+\sigma+\alpha\theta}{\sigma(1+\theta)}$, $\tilde{\lambda} = \left(\lambda E^{\frac{\theta}{\sigma}}\right)^{\frac{1}{1+\theta}}$, and

$$\tilde{K}_{ij}^A = \beta^{\frac{\theta}{\sigma(1+\theta)}} \bar{A}_i t_{ij}^{-\mu\tilde{\epsilon}} \left(\sum_k K_{ik}^{L_E} \left(H_k^F\right)^{\frac{\theta(\alpha-1)}{\sigma}}\right)^{-\frac{1}{1+\theta}}$$

is exogenous under $\mathcal{F}_1$ (in which $K_{ij}^{L_E} = \alpha^\theta e_{ij}^w e_{ij}^A \bar{u}_j^\theta \left(H_i^F\right)^{(1-\alpha)\theta} d_{ij}^{-\theta}$).

City in the circle

To solve the analytic solution of l_i in the circle, it is equivalent to solve the following equation

$$y(x) = \int_{-\pi}^{\pi} C\cos^2\left(\frac{x-s}{2}\right) y(s)^{\kappa}\, ds$$

$$y'(x) = -\int_{-\pi}^{\pi} \frac{1}{2} C\sin(x-s)\, y(s)^{\kappa}\, ds$$

$$\begin{aligned} y''(x) &= -\int_{-\pi}^{\pi} \frac{1}{2}\cos(x-s)\, y(s)^{\kappa}\, ds \\ &= -\int_{-\pi}^{\pi} \left(\cos^2\left(\frac{x-s}{2}\right) - \frac{1}{2}\right) y(s)^{\kappa}\, ds \\ &= -y + Y \end{aligned}$$

where $Y = \tau \int_{-\pi}^{\pi} y(s)^{\kappa}\, ds$ a constant. Like above it is equivalent with equation

$$ww'_y = -y + Y \Longrightarrow$$

$$\frac{w^2}{2} = -\frac{y^2}{2} + Yy + \frac{C_1}{2} = -\frac{(y-Y)^2}{2} + \frac{C_1 + Y^2}{2} \Longrightarrow$$

where C_1 is a constant.

$$y'_x = \pm\sqrt{-y^2 + 2Yy + C_1} \Longrightarrow$$

For the time being only consider the case of $y'_x \geq 0$

$$\int \frac{1}{\sqrt{-y^2 + 2Yy + C_1}} dy = x + C_2 \Longrightarrow$$

$$-\sin^{-1}\frac{-2y + 2Y}{\sqrt{4Y^2 + 4C_1}} = x + C_2 \Longrightarrow$$

(see 5.4.13 of book Zwillinger, Daniel, ed. CRC standard mathematical tables and formulae. CRC press, 2002.)

$$y = Y + \sqrt{Y^2 + C_1}\sin(x + C_2)$$

We show that when $0 \leq |\kappa| \leq 1$ (it is the same for $-1 \leq \kappa < 0$), $Y_{max} = Y_{min}$ i.e. $y = Y$ a constant. Obviously, the maximum of y is $Y_{max} = Y + \sqrt{Y^2 + C_1}$ and minimum is $Y_{min} =$

$Y - \sqrt{Y^2 + C_1}$. Notice that $Y_{max} \leq Y_{max}^{\kappa} \max \int_{-\pi}^{\pi} \tau \cos^2\left(\frac{x-s}{2}\right) ds$ and $\int_{-\pi}^{\pi} \tau \cos^2\left(\frac{x-s}{2}\right) ds = \pi\tau$thus $Y_{max}^{1-\kappa} \leq \pi\tau$, also $Y_{min}^{\kappa} \min_x \int_{-\pi}^{\pi} \tau \cos^2\left(\frac{x-s}{2}\right) ds \leq Y_{min}$ i.e. $\pi\tau \leq Y_{min}^{1-\kappa}$. Thus, it has to be that $Y_{min} = Y_{max}$ i.e. $\sqrt{Y^2 + C_1}$.

City in the infinite line

To solve the analytic solution of l_i in the infinite line, it is equivalent to solve the following equation

$$y(x) = \int_{-\infty}^{\infty} C \exp\left(-\gamma (x-s)^2\right) y(s)^{\kappa} ds$$

Guess the solution is the form of $y(x) = b \exp\left(-ax^2\right)$ where $a > 0$. Substitute it into above equation,

$$\begin{aligned}
b \exp\left(-ax^2\right) &= b^{\kappa} \int_{-\infty}^{\infty} C \exp\left(-\gamma (x-s)^2\right) \exp\left(-a\kappa s^2\right) ds \\
&= b^{\kappa} \int_{-\infty}^{\infty} C \exp\left(-(a\kappa+\gamma) s^2 + 2\gamma x s - \gamma x^2\right) ds \\
&= b^{\kappa} \int_{-\infty}^{\infty} C \exp\left(-(a\kappa+\gamma)\left(s - \frac{\tau}{a\kappa+\tau} x\right)^2 + \left(\frac{\tau^2}{a\kappa+\tau} - \tau\right) x^2\right) ds \Rightarrow
\end{aligned}$$

$$b^{\kappa-1} \exp\left(\left(\gamma - \frac{\gamma^2}{a\kappa+\gamma} - a\right) x^2\right) = C \int_{-\infty}^{\infty} \exp\left(-(a\kappa+\gamma) s^2\right) ds \Rightarrow$$

$$a = \frac{\gamma(\kappa-1)}{\kappa}.$$

B Data Appendix

B.1 Constructing Time Use Data

Using the Census (2011) we find that the average commuting time from and to work for a U.S. citizen is about 50 minutes. We divide this by two to account for days off-work and unemployed people. The BLS (2013) times spend in working, work-related meetings and education, and leisure and civic/religious activities. Working and work-related activities (e.g. business meetings, time spend playing gold with clients etc) include required commuting time (but do not report the specific) so we subtract the commuting times off these two uses split according to the relative fraction of time they represent.[16] We use the reminder of the working time as the working time of an average agent in the model. The remaining work-related activities spent together with time spent on education gives us the time spend for productivity improvement. Finally, we consider as leisure time and civic/religious activities as time spent in leisure in the model.

[16]We exclude the other categories as we do not model time for shopping and we consider personal care and sleep as taking care of basic indispensable needs.