An Introduction to Dynamic General Equilibrium: Modeling and Simulation

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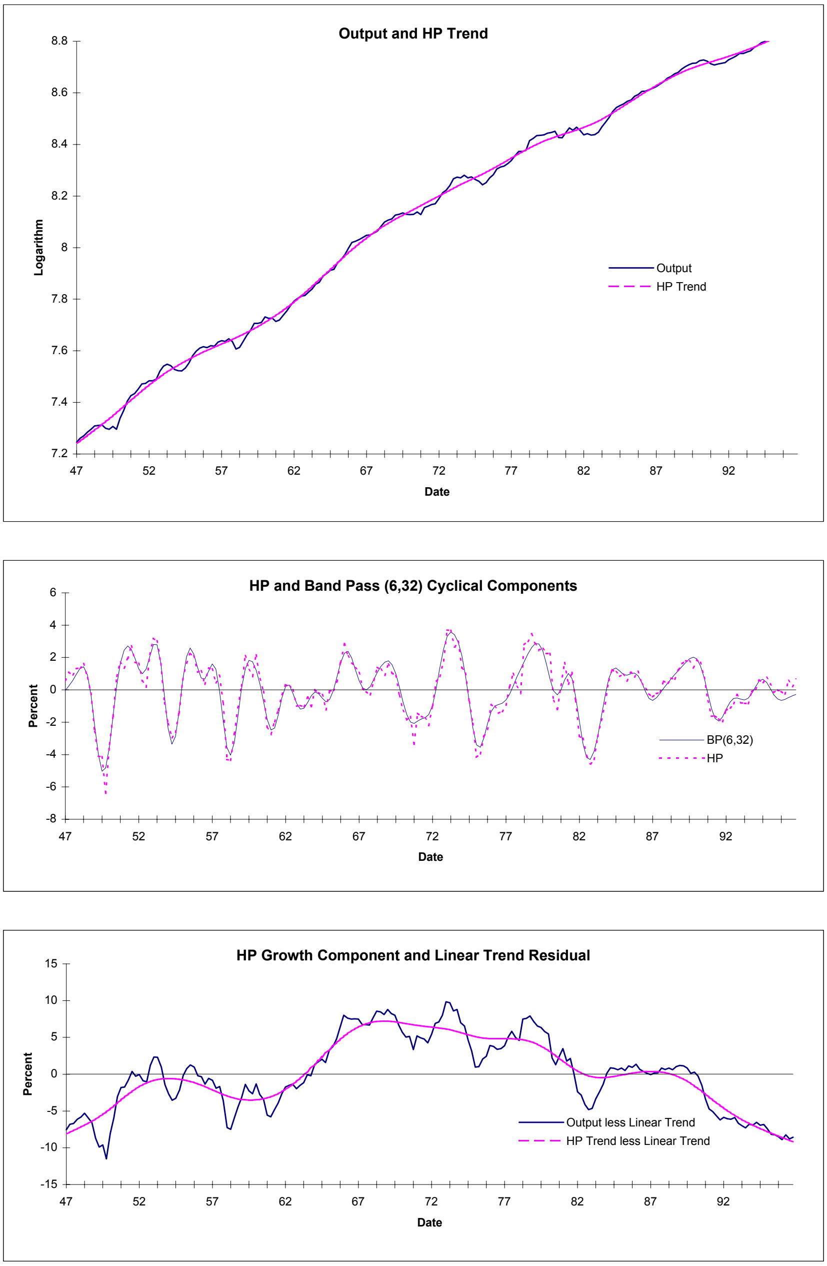
Summer 2014

The material discussed here builds upon the King-Rebelo article assigned in the course outline. It is closely related and has more details on certain aspects than I have time to cover. You should read it in conjunction with these notes. Other readings are mentioned within the notes.

1 Data Detrending and Stylized Facts

- Many real macroeconomic quantities grow over time (tech. progress?) but also show fluctuations around this trend (e.g. see Figure 1 in King and Rebelo).

Figure 1



Note: Sample period is 1947:1 - 1996:4.

- Helpful to break this into long run and short run components and study them separately: growth and cycles.

$$y_t = y_t^c + y_t^g$$

- HOW? Detrending via log-linear regressions or HP Filter or others.

$$\ln Y_t = \beta_1 + \beta_2 \times t + e_t$$

$$y_t^g = \hat{\beta}_1 + \hat{\beta}_2 \times t$$

$$y_t^c = \hat{e}_t$$

- Hodrick-Prescott Filter (HP filter)
  - the HP filter identifies a trend that fluctuates slightly overtime.

– more specifically, the trend component (y_t^g) is chosen to minimize a loss function.

$$\min_{\{y_t^g\}_{t=0}^{\infty}} \sum_{t=1}^{\infty} \left\{(y_t - y_t^g)^2 + \lambda[(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)]^2\right\}$$

– as $\lambda \to \infty$, the trend component approaches a linear trend.

– Lets look at how we can do the same for Indian real gdp using MATLAB.

1.1 Stylized Facts

- Business cycle statistics for the U.S. economy (see K+R table 1)

Table 1
Business Cycle Statistics for the U.S. Economy

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

Note: All variables are in logarithms (with the exception of the real interest rate) and have been detrended with the HP filter. Data sources are described in Stock and Watson [1998], who created the real rate using VAR inflation expectations. Our notation in this table corresponds to that in the text, so that Y is per capita output, C is per capita consumption, I is per capita investment, N is per capita hours, w is the real wage (compensation per hour), r is the real interest rate, and A is total factor productivity.

from kingrebelo.pdf

1.1.1 Volatility

- consumption, wages, labour productivity, capital stock and TFP are less volatile than output;

- investment and capital utilization are more volatile than output;

- total hours worked and employment are about as volatile as output;

- hours per worker are much less volatile than output.

Persistence

- First-order autocorrelations are large and positive.

1.1.2 Cyclicality

-
  - strongly procyclical: consumption, investment, total hours worked, labour productivity and TFP. Imports are more procyclical than exports.
  - acyclical: wages, capital stock and government expenditures
  - countercyclical: real interest rate

1.1.3 Cycles in Emerging Economies

Table 1
Business cycles in emerging and developed economies

(a) Standard deviations

	% Standard deviation			% Standard deviation / % Standard deviation of GDP				
	GDP	*R*	NX	PC	TC	INV	EMP	HRS
Emerging economies								
Argentina	4.22	3.87	1.42	1.08	1.17	2.95	0.39	0.57
	(0.36)	(0.52)	(0.11)	(0.05)	(0.03)	(0.13)	(0.07)	(0.08)
Brazil	1.76	2.34	1.40	1.93	1.24	3.05	0.89	1.95
	(0.23)	(0.26)	(0.45)	(0.38)	(0.23)	(0.26)	(0.13)	(0.33)
Korea	3.54	1.42	3.58	1.34	2.05	2.20	0.59	0.71
	(0.50)	(0.23)	(0.55)	(0.07)	(0.18)	(0.16)	(0.07)	(0.05)
Mexico	2.98	2.64	2.27	1.21	1.29	3.83	0.43	0.33
	(0.36)	(0.38)	(0.28)	(0.08)	(0.06)	(0.17)	(0.09)	(0.08)
Philippines	1.44	1.33	3.31	0.93	2.78	4.44	1.34	NA
	(0.17)	(0.13)	(0.45)	(0.11)	(0.44)	(0.43)	(0.33)	
Average	2.79	2.32	2.40	1.30	1.71	3.29	0.73	0.89
Developed economies								
Australia	1.19	2.00	1.02	0.84	1.20	4.13	1.13	1.40
	(0.09)	(0.17)	(0.08)	(0.07)	(0.08)	(0.22)	(0.10)	(0.14)
Canada	1.39	1.54	0.76	0.74	0.84	2.91	0.75	0.82
	(0.08)	(0.12)	(0.06)	(0.05)	(0.05)	(0.18)	(0.04)	(0.04)
Netherlands	0.93	0.93	0.67	1.17	1.44	2.66	1.27	NA
	(0.06)	(0.12)	(0.07)	(0.08)	(0.12)	(0.22)	(0.14)	
New Zealand	1.99	1.92	1.31	0.82	0.86	3.32	1.15	1.28
	(0.18)	(0.19)	(0.13)	(0.08)	(0.09)	(0.34)	(0.10)	(0.12)
Sweden	1.35	1.92	0.86	1.01	1.67	4.18	1.24	2.94
	(0.14)	(0.26)	(0.09)	(0.10)	(0.22)	(0.34)	(0.13)	(0.17)
Average	1.37	1.66	0.92	0.92	1.08	3.44	1.11	1.61

(b) Correlations with GDP

	R	NX	PC	TC	INV	EMP	HRS
Emerging economies							
Argentina	−0.63	−0.89	0.94	0.97	0.94	0.36	0.52
	(0.08)	(0.02)	(0.11)	(0.01)	(0.01)	(0.11)	(0.11)
Brazil	−0.38	−0.03	0.48	0.58	0.80	0.62	0.75
	(0.22)	(0.18)	(0.16)	(0.19)	(0.08)	(0.15)	(0.09)
Korea	−0.70	−0.86	0.96	0.92	0.94	0.91	0.96
	(0.11)	(0.04)	(0.02)	(0.03)	(0.02)	(0.04)	(0.02)
Mexico	−0.49	−0.87	0.93	0.96	0.96	0.56	0.37
	(0.13)	(0.05)	(0.02)	(0.02)	(0.02)	(0.13)	(0.13)
Philippines	−0.53	−0.40	0.69	0.51	0.76	0.26	NA
	(0.12)	(0.14)	(0.09)	(0.11)	(0.10)	(0.20)	
Average	−0.55	−0.61	0.80	0.79	0.88	0.54	0.65

neumeyer-perri-jme2005.pdf

- Business cycle statistics for emerging economies (see Neumeyer and Perri (2005) table 1)
- Consumption more volatile than output; output more volatile than developed countries
- interest rates and net exports: strongly counter-cyclical
- Business cycle statistics for India (see Ghate, Pandey, and Patnaik (2011) table 3 and 5), similar.

1.1.4 Stylized Facts of Economics Growth

Balanced Growth

- output grows at a (more or less) constant rate;
- capital grows faster than labor input;
- growth rates of output and capital stock are about the same;
- the rate of profit on capital has no trend.
- the ratio of labour income to output has no trend;
- the ratio of investment to output has no trend;

- the ratio of consumption to output has a small (not statistically significant) positive trend.

2 Model

- Looking for a simple structure that can capture basic features of aggregate economic data.
- So ignore that agents are different. Let one representative agent decide everything for everyone. (Planner's problem).
- Focus on consumption-saving decision and labour-leisure decision for consumer-workers. This is how capital is accumulated and total hours worked is determined.
- Basically think of this as solow growth model with endogenous savings decision and endogenous labour supply. And shocks to cause fluctuations around the growth path of the economy.
- For now, no markets, no prices. We will add realism one by one until we run out of time!
- With identical agents there is no material difference between one and many agents. The planner will make everyone behave the same so you could just focus on one agent and save notation.

- So lets start with a large number, M, of infinitely lived agents who have identical expected utility.
- Let C be total consumption and $c = C/M$ be per capita consumption

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_{t,} L_t), \ \beta > 0, 0 < \beta < 1 \tag{1}$$

- c_t is per capita consumption in period t, and L_t is leisure in period t, while β is the discount factor.
- This maximization of life-time discounted utility is based on information in time zero.
- Since M is a constant population size lets just normalized to unity. Could allow a growing population and then detrend all variables to per capita terms. Then population growth rate would be a parameter of the model.
- U(.) is concave and increasing in c and L

  Two commonly used specifications:

$$U(c, L) = \ln c_t + x \ln L_t$$

and

$$\frac{1}{1-\sigma}\left[c_t v(L_t)\right]^{1-\sigma} - \frac{1}{1-\sigma}, \sigma > 0, \sigma \neq 1$$

- See appendix in King and Rebelo 2000 for a discussion or KPR 1988.

  These preferences imply agents will want to "smooth consumption and leisure"

  Also these imply "intertemporal substitution of c and L" as the costs of c and L vary over time.

- Endowments of representative agent: One unit of time. Spent on working or on leisure :

$$N_t + L_t = 1 \qquad (2)$$

- In decentralized version agent works for a firm and earns a wage in a competitive labour market. In the planner's problem, the agent owns the technology for producing the single good and the planner allocates how much the agent must work. Without imperfections, the welfare theorems hold so that the two give rise to identical allocations.
- Production technology traditionally uses two inputs: Physical Capital and Labour:

$$Y_t = A_t F(K_t, N_t) \quad (3)$$

- Many others added in the literature: variable utilization of capital and labour(Burnside Eichenbaum and Rebelo 1993), intangible inputs like human capital, knowledge capital and organizational capital (Cooper and Johri (2002)). Also see Johri and Letendre (2007) for estimation, discussion and comparison of some of these. (Discuss if time permits)

- Today we will ignore labour augmenting technical progress to focus on business cycle issues. Simply detrend all growing variables as shown in King-Rebelo to get to our starting point.

- Assumptions and Restrictions

  $A_t > 0$ for all t, $\quad K_0 > 0$

- A_t is a random shock to total factor productivity

- $F(.)$ is twice continously differentiable concave and homogenous of degree one.

  F satisfies the Inada Condition:

- $lim_{K \to \infty} F_K = 0$ and

- $lim_{K \to 0} F_K = \infty$

- Note this is a single good economy, which can be used for consumption or investment. Relaxing this is easy. The international macro literature often has multiple goods. Often models have consumption and investment goods produced in different sectors.

- It is common to have a government that consumes some of the goods produced in the economy, so our resource constraint is :

$$Y_t = c_t + G_t + I_t \qquad (5)$$

- Stock of capital evolves according to

$$K_{t+1} = I_t + (1 - \delta)K_t \qquad (6)$$

δ is the rate of depreciation. $\quad 0 < \delta < 1.$

- In the planner problem, the planner allocates goods to consumption by agents, by government and to investment. In the decentralized version, either the worker-consumer can invest and accumulate capital or the firm can do it. [Note we have assumed I=S without the need for intermediation or any financial products. Relax if time permits.]

- The planner's optimal control problem:

$$max E \sum_{t=0}^{\infty} \beta^t u(c_t, L_t) \qquad (7)$$

subject to (2) to (6)

- (3), (5) and (6) can be combined to get:

$$c_t + G_t + K_{t+1} = A_t F(K_t, N_{t)} + (1 - \delta)K_t \ (8).$$

- The optimal path of capital accumulation can be found by choosing sequences for

$$\{c_t\}_0^\infty, \{L_t\}_{t=0}^\infty, \{N_t\}_{t=0}^\infty \text{ and} \{K_{t+1}\}_{t=0}^\infty$$

to max (7) s.t. (8) and (2)

- Write the Langrangean as:

$$\begin{aligned} L = E[\quad & \sum_{t=0}^{\infty} \beta^t u(c_t, Lt) + \sum_{t=0}^{\infty} \beta^t \lambda_t \\ & [AF(K_t, N_t) + (1-\delta)K_t - c_t - G_t - K_{t+1}] \\ & + \sum_{t=0}^{\infty} \beta^t \omega_t \left[1 - L_{t-}N_t\right]] \end{aligned}$$

- F.O.C.

$$c_t : U_{c_t} = \lambda_t \qquad (9)$$

$$L_t : U_{L_t} = \omega_t \qquad (10)$$

$$N_t : \lambda_t A_t F_{N_t} = \omega_t \ (11)$$

$$K_{t+1} : \beta E \lambda_{t+1} \left[A_{t+1} F_{K_{t+1}} + 1 - \delta\right] = \lambda_t \ (12)$$

- Remember these conditions hold for all time periods 't'.
- An optimal program will satisfy these FOC, the intial conditions and the original constraints as well as the transversality condition:

$$lim_{t\to\infty}\beta^t\lambda_t K_t = 0.$$

- Intutively, what does this restriction mean ? Mathematically what does it do ?
- Note that the choices of c_t, N_t and K_t are jointly made for all t.
- The FOCs together with the constraints faced by the planner constitute the system of equations which together determine the equilibrium sequence of endogenous variables.
- The optimal decisions respect the resource constraints of the economy. Why? Because take into account the shadow prices w_t and λ_t associated with our two constraints.
- So also solve for w_t and λ_t.
- Alternatively you can solve model using dynamic programming but we will skip that.
- Basic equivalence of this problem with the competitive market outcome.

2.1 A Decentralized Model

- A single firm running the aggregate production function. Or many atomistic firms of measure one.

- Large number of identical households of measure one.

- I will have 1 firm and 1 consumer worker (household) to save notation and having to put in adding up constraints later (does not affect the economics of system since identical). Markets are competitive.

- Households own the firm and the capital stock. Earn any profits and rental income

- Households rent capital to the firm (rental rate is r_t^k) and supply labour to the firm (wage rate is $\tilde{w}_t$).

- Firms problem is easy since they accumulate no state variables so just solve single period problem each period

$$\max_{\tilde{K}_t, N_t} \tilde{\Pi}_t = A\ \tilde{K}_t^{\alpha}\ N_t^{1-\alpha} - r_t^k \tilde{K}_t - \tilde{w}_t N_t$$

FOC's imply

$$r_t^k = \alpha\ \frac{\tilde{Y}_t}{\tilde{K}_t}, \qquad \tilde{w}_t = (1-\alpha)\ \frac{\tilde{Y}_t}{N_t}$$

Note, factor prices depend on the aggregate state:

$$\tilde{w}_t = \tilde{w}(\tilde{K}_t), \qquad r_t^k = r^k(\tilde{K}_t)$$

- Representative household's problem taking prices as given,
  - Household chooses $\tilde{c}_t$, n_t, $\tilde{\imath}_t$, $\tilde{k}_{t+1}$

$$maxE \sum_{t=0}^{\infty} \beta^t u(c_t, L_t)$$

subject to

$$\tilde{c}_t + \tilde{I}_t = r^k \ \tilde{K}_t + \tilde{w} \ n_t + \tilde{\pi}_t - T_t$$

$$\tilde{K}_{t+1} = (1 - \delta)\tilde{K}_t + \tilde{I}_t$$

  - Model the government simply as running a balanced budget

$$G_t = T_t$$

Easy to introduce deficit financing and distortionary taxes.

– Example: First order conditions assuming log-log utility

$$U(c, L) = logc_t + \chi_t \log L_t$$

$\Rightarrow$

$$U_c = \frac{1}{c_t}, U_L = \frac{\chi_t}{L_t}$$

So (9) $\Rightarrow$

$$\lambda_t = \frac{1}{c_t}$$

$$\frac{1}{\tilde{c}_t}\tilde{w}(\tilde{K}_t) = \frac{\chi_t}{1 - n_t} \qquad (n_t)$$

$$(\gamma)\frac{1}{\tilde{c}_t} = \beta \; \frac{1}{\tilde{c}_{t+1}} \left[r^k(\tilde{K}_{t+1}) + 1 - \delta \right] \qquad (\tilde{k}_{t+1})$$

– Two things to note. γ is assumed to be one in our notes but if there is labour augmenting technical progress then it would show up as in King-Rebelo. Our FOCs can be solved to get labour and capital supply functions.

– Using factor prices from firm's problem and market clearing in factor markets, the FOCs above become

$$\frac{1}{\tilde{C}_t}(1-\alpha)\frac{\tilde{Y}_t}{N_t} = \frac{\chi_t}{1-N_t}$$

$$\frac{1}{\tilde{C}_t} = \beta\,\frac{1}{\tilde{C}_{t+1}}\left[\alpha\frac{\tilde{Y}_{t+1}}{\tilde{K}_{t+1}} + 1 - \delta\right]$$

and the budget constraint becomes

$$\tilde{C}_t + \tilde{I}_t = \alpha\frac{\tilde{Y}_t}{\tilde{K}_t}\tilde{K}_t + (1-\alpha)\frac{\tilde{Y}_t}{N_t}N_t + 0 - T_t$$

$$\Leftrightarrow \tilde{C}_t + \tilde{I}_t = \tilde{Y}_t - G_t$$

– The equilibrium system of equations implied by the decentralized model is identical to the system of equations derived from the planner's problem.

2.1.1 The steady State

– Since we have already detrended the model, none of the variables are growing. One potential solution to our system of equations is a steady state: c, N, K, I, Y etc are all constant for all t.

– Lets solve for the steady state by replacing the variables in above equations with their steady state values (no time subscripts)
  so we have

$$\frac{1}{\tilde{C}}(1-\alpha)\frac{\tilde{Y}}{N} = \frac{\chi}{1-N} \qquad (SS1)$$

$$\frac{1}{\tilde{C}} = \beta\ \frac{1}{\tilde{C}}\left[\alpha\frac{\tilde{Y}}{\tilde{K}} + 1 - \delta\right] \qquad (SS2)$$

– Lets manipulate SS2 to solve for the capital output ratio of the economy (cancel steady state consumption and solve

$$(1/\beta - (1-\delta))\frac{1}{\alpha} = \frac{1}{K/Y}$$

– Now using capital accumulation equation in steady state

$$K = (1-\delta)K + I$$

  we get $I = \delta K$

– Plug into resource constraint $Y = C + I + G$ to get $Y = C + \delta K + G$

- Now divide by Y, pick a value for SS govt. spending as a fraction of Y, (say .2 for US data) to get consumption to output ratio

$$C/Y = 1 - .2 - \delta K/Y$$

- Now manipulate (SS1) to write hours as a function of C/Y

$$\frac{1}{(1-\alpha)}\frac{\tilde{C}}{Y} = \frac{1-N}{\chi N}$$

- There are two possibilities now. Look at data to calculate st. st. hours (.3 in US data) and plug into above equation and solve for χ or vice versa. I suggest former.
- Next solve for Y from production function. Normalize St. St. productivity to unity $A = 1$.

$$Y = K^{\alpha} N^{1-\alpha} = K^{\alpha}.3^{1-\alpha}$$

Rewrite as

$$Y^{1-\alpha} = (K/Y)^{\alpha}.3^{1-\alpha}$$

- Solve for Y, then use Y to get all the other values viz. $K = Y\frac{K}{Y}$ etc.

– There exists a unique steady state (SS). Typically we study behaviour of macro variables as percent deviations from their SS value. [Recall model was detrended as was the data so both model quantities and data have an underlying trend.]

2.2 TRANSITIONAL DYNAMICS (out of steady state behaviour)

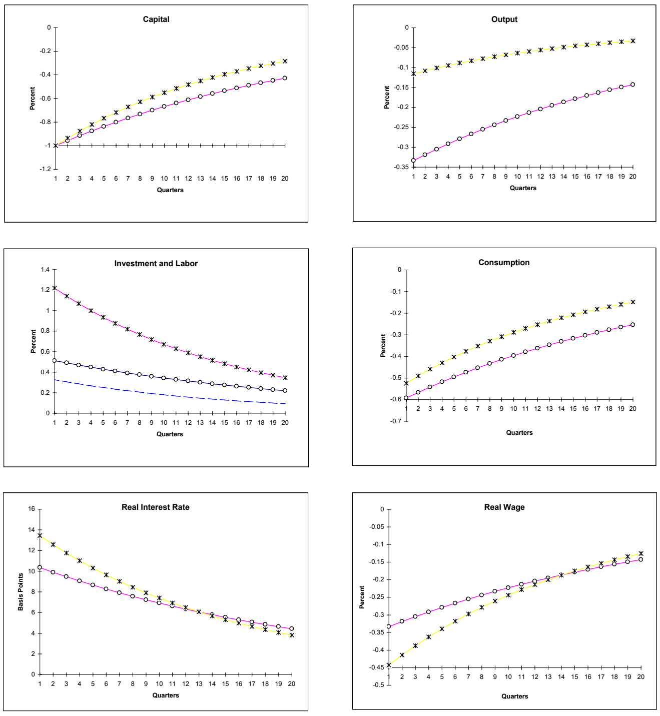
Look at figure 6 in King-Rebelo (2000). All variables are in % deviation from their corresponding SS values.

– Example:

$$\hat{K}_t = \frac{K_t - K^*}{K^*}$$

– The capital stock is 1% below SS level.

Figure 6



Note: Basic RBC model (stars); fixed labor model (circles); dashes in panel 3 represent the labor response.

from kingrebelo-5.pdf

– Since K is low, individuals work harder to make up lost output.

– This output is put towards capital accumulation so C is below ss and I is above ss.

– Note that below ss K implies high MP_K and high real rate of return on investing.

$\Rightarrow$ High real interest rate in market

$\Rightarrow$ High return acts as an "allocative signal" to postpone consumption.

Why?

Our Euler Equation derived earlier is

$$\frac{c_{t+1}}{\beta c_t} = [A_{t+1} F_{K_{t+1}} + 1 - \delta]$$

– when K_{t+1} is low relative to SS.$\Rightarrow$ RHS is high so that c_t is low relative to c_{t+1}

Since K_t monotonically rises to reach its SS level, c will also monotonically rise to reach SS level.

$\Rightarrow$ c is initially below SS.

– Leisure behaves like consumption because it too is driven by the high return on capital.

– Transitional dynamics helpful in understanding the model but not the full story. Also helpful for analyzing impulse responses when the model is shocked.

2.3 Introducing Shocks

- Many different sources of fluctuations have been explored in the DGE literature. We begin with the original (and very potent) shock to TFP.
- Use the solow residual to identify the stochastic component of TFP.
- Solow residual (SR) measured as :

$$\ln SR_t = \ln Y_t - (1-\alpha)\ln N_t - \alpha \ln K_t$$

If the theoretical model is correct,

$$\ln SR_t = \ln A_t + (1-\alpha)\ln(\gamma) \times t.$$

Therefore, given values for α and γ, the productivity shocks are calculated as

$$\ln A_t = \ln SR_t - (1-\alpha)\ln(\gamma) \times t.$$

- We can feed these stationary productivity shocks into the model and calculate how all variables respond. The shocks follow an AR(1) process in logs

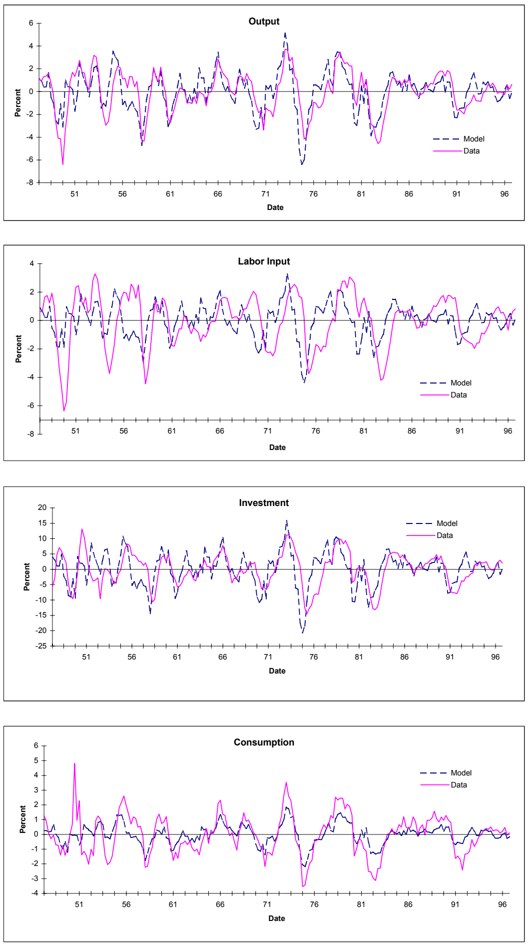
$$\ln A_{t+1} = \rho \ln A_t + \varepsilon_{t+1}$$

$$0 < \rho < 1, \qquad \varepsilon_{t+1} \sim iid(0, \sigma_\varepsilon^2)$$

2.3.1 Predictions of the Model with TFP shocks

– K+R figure 7 compares historical and simulated paths for the U.S. economy.

Figure 7



Note: Sample period is 1947:2 - 1996:4. All variables are detrended using the Hodrick-Prescott filter.

from kingrebelo.pdf

– Moment matching: compare K+R tables 1 and 3.

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from kingrebelo.pdf

Table 3
Business Cycle Statistics for Basic RBC Model[35]

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.39	1.00	0.72	1.00
C	0.61	0.44	0.79	0.94
I	4.09	2.95	0.71	0.99
N	0.67	0.48	0.71	0.97
Y/N	0.75	0.54	0.76	0.98
w	0.75	0.54	0.76	0.98
r	0.05	0.04	0.71	0.95
A	0.94	0.68	0.72	1.00

Note: All variables have been logged (with the exception of the real interest rate)and detrended with the HP filter.

from kingrebelo.pdf

– Internal propagation of shocks

Autocorrelation	Model	Data
Prod. shocks (A)	0.72	0.74
Output (Y)	0.72	0.84

– Impulse response functions: see K+R figures 9 and 10.

2.4 other shocks

– Christiano and Eichenbaum (1992) added government spending shocks.

– govt. consumption evolves according to $G_t = \gamma^t g_t$, where

$$ln(g_t) = (1 - \rho_g)\ln(g) + \rho_g \ln(g_{t-1}) + \epsilon_{gt}.$$

$$0 < \rho_g < 1, \qquad \epsilon_{gt} \sim iid(0, \sigma_g^2)$$

– Display and discuss impulse response to G-shocks.

– Many studies include shocks to preferences which is tantamount to shocking the MRS between consumption and leisure. Help to improve the fit of the model to total hours series.

– An example with preference shocks, imperfect competition and organizational capital is Clarke and Johri (2009) which contains many references to related papers.

– χ_t is a preference shock where
$ln(\chi_t) = (1-\rho_b)\ln(b) + \rho_b \ln(\chi_{t-1}) + \epsilon_{bt}$.

$$0 < \rho_b < 1, \qquad \epsilon_{bt} \sim iid(0, \sigma_b^2)$$

– The labour market in the standard RBC model

– labour demand comes from firm's FONCs

$$\tilde{w}_t = (1-\alpha)\frac{A_t\tilde{K}_t^\alpha}{N_t^\alpha} \Rightarrow N_t^d = \left[(1-\alpha)\frac{A_t\tilde{K}_t^\alpha}{\tilde{w}_t}\right]^{\frac{1}{\alpha}}$$

– labour supply comes from the representative household's FOCs

$$\tilde{w}_t = \frac{\chi_t\tilde{C}_t}{1-N_t} \Rightarrow N_t^s = 1 - \frac{\chi_t\tilde{C}_t}{\tilde{w}_t}$$

– do graphical representation. TFP shocks shift both demand and supply of labour. Preference shocks shift only labour supply. Similarly government shocks shift labour supply.

2.5 Main Criticisms of the RBC Model

I. Labour Market

– A few statistics

1. $corr(w, Y)$ is too high
2. $corr(w, N)$ is too high
3. $var(N)/var(w)$ is too small
4. In the model, the labour supply elasticity is much larger than in the data

II. Productivity Shocks

1. Hard to identify the macro shocks that produce the productivity variations suggested by the Solow residuals.

2. Solow residuals often decline, suggesting recessions are caused by technological regress.

3. Measurement errors: SR do no take into account of

variable labour effort and variable capital utilization rates.

2.6 Introducing variable utilization of capital

- We don't have to but let's be realistic and allow the firm to accumulate physical capital.
- Let the firm be owned by the household/consumer-worker and she receives any profits earned by the firm.
- The firm produces output using

$$Y_t = N_t^{\alpha} \tilde{K}_t^{1-\alpha}, \tag{2}$$

where $\tilde{K}_t$ refers to capital services defined as

$$\tilde{K}_t = u_t K_t, \tag{3}$$

where K_t is the firm's stock of physical capital and u_t is the utilization rate of that capital.

- The firm accumulates capital according to

$$K_{t+1} = [1 - \delta(u_t)]K_t + I_t, \tag{4}$$

- Note $\delta(\cdot)$ implies increasing capacity utilization causes increased depreciation of capital.

- $\delta(\cdot)$ satisfies the conditions $\delta'(\cdot) > 0$, $\delta''(\cdot) \geq 0$.
- The firm chooses sequences of N_t, $\tilde{K}_t$ I_t to maximize current and expected future profits, given by

  Each period, the firm pays out any profits earned to the household that owns it. Profits are given by

$$\Pi_t = Y_t - w_t N_t - I_t \tag{5}$$

- The firm chooses sequences of N_t, $\tilde{K}_t$, and I_t to maximize current and expected future profits,

$$\sum_{s=0}^{\infty} \beta^{t+s} \frac{\lambda_{1t+s}}{\lambda_{1t}} \Pi_{t+s} \tag{6}$$

  where $\beta^{t+s}\frac{\lambda_{1t+s}}{\lambda_{1t}}$ is the household-owner's stochastic discount factor.
- We can focus on any representative pair of time periods: lets pick t and t+1. So s=0 and s=1 in this case.
- The relevant parts of our lifetime sum are:

$$\beta^t \frac{\lambda_{1t}}{\lambda_{1t}} \Pi_t + \beta^{t+1} \frac{\lambda_{1t+1}}{\lambda_{1t}} \Pi_{t+1} \tag{7}$$

- Sub in for profits and replace I using capital accumulation equation and maximize over N_t, u_t, K_{t+1}

- The firm's first-order conditions are then

$$w_t = \alpha \frac{Y_t}{N_t} \tag{8}$$

$$\delta'(u_t) K_t = (1-\alpha) \frac{Y_t}{u_t} \tag{9}$$

$$1 = \beta E_t \frac{\lambda_{1t+1}}{\lambda_{1t}} [(1-\alpha) \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta(u_{t+1})]. \tag{10}$$

- Lets linearize (9) to interpret what utilization buys us in the model and substitute into linearized production function (on board)

- Elasticity of depreciation (ξ) wrt utilization matters a lot! As $\xi \to 0$ the response of output to labour increases. As $\xi \to \infty$ we are back to constant utilization case.

- As $\xi \to 0$, the labour demand curve becomes flat and responds only to tech shocks. (On board). So shifts in labour supply have big effects on hours and little effects on wages (like data).

- Utilization also helps on the size of shocks. Recall that we measured the TFP shock from the detrended solow residual. With utilization present, we mis-measured the true shock: the solow residual has the shock and the utilization present in it.

- Correcting for utilization substantially reduces the volatility of the modified solow residual. Also substantially reduces the likelihood of technological regress.
- Utilization also acts like a wedge in the euler equation since we have time-varying depreciation.

2.7 Indivisible labour and the Extensive Margin

- Hansen and Wright point out in 1992 article:
  - 1. Hours are less variable than Y or approximately the same
    $$\frac{\sigma_h}{\sigma_Y} \approx 0.78 - 1.01$$
  - Generally establishment survey has higher volatility of hours.
  - 2. Hours fluctuate more than productivity $= (Y/h)$
    $$\frac{\sigma_h}{\sigma_{alp}} \approx 1.37 \text{ to } 2.15$$
  - 3. Correlation between hours and productivity is near zero.
    $$Cor(h, alp) \approx -.35 \text{ to } .10$$
- However the baseline model with log-log preferences:

  1. Model predicts $\sigma_y = 1.30$

     US data $\sigma_y = 1.92$

2. Model predicts $\frac{\sigma_h}{\sigma_w} = .94$

   Data $\frac{\sigma_h}{\sigma_{alp}} > 1$

3. Model predicts $Cor(h, w) = 0.93$

   Data $Cor(h, w) \approx 0$

- It seems that labour supply elasticity is not high enough to get hours to move in response to wage changes.

- Indivisible labour model makes elasticity ∞ and helps on 1 and 2.

  σ_y increases to 1.73

  $\frac{\sigma_h}{\sigma_y}$ increases from .49 to .76

  $\frac{\sigma_h}{\sigma_w}$ is $2.63, Cor(h, w) = .76$

- How can we increase labour supply elasticity? In fact, assumed elasticity is already too high relative to micro studies.

- Note that when $A_t \uparrow \Rightarrow$ wages increase. This has two effects on C_t and on N_t.

  a. Substitution effect: $\uparrow N$ and C

  b. Income Effect: $\downarrow N$ and $\uparrow C$

     So change is ambiguous

- The degree of persistence matters because income effect becomes bigger as shock lasts longer.

- King and Rebelo (2000) show that for the log-log case $\hat{N} = \frac{1-N}{N\eta}(\hat{W} + \hat{\lambda})$
- Recall λ is the shadow price of an extra unit of consumption or the wealth effect. So elasticity of pure substitution effect is $\frac{1-N}{N\eta}$
- log utility $\Rightarrow \eta = 1$ (eq. 4.2 in K-R gives preferences)

  N is fraction of time spent working

  K-R use $N = .2 \Rightarrow$ elasticity $= 4$

  others use $N = .33 \Rightarrow$ elasticity $= 2$

  However micro evidence $\Rightarrow$labour supply elasticity < 1
- At this level $\sigma_n = .33$ (v. low)

  This is a common problem not unique to RBC models.
- HANSEN (1985) based on Rogerson (1988) explains how to reconcile low micro elasticities with high macro elasticities
- KEY IDEA: introduce the extensive margin. Total hours vary not only because households choose to work more or less hours as wages flucutuate.
- More important source of hours variation is movements into and out of jobs.

  $var(\log H_t) = var(\log h_t) + var(\log N_t) + 2cov(logh_t, \log N_t)$

H_t is the total hours

N_t is number of individuals working

h_t is average hours worked

- Variance decomposition of H_t :

  55% due to N_t

  20% due to h_t

  25% due to Cov.

- The basic RBC model assumes all variation due to h_t. Hansen's indivisible labour model will assume all is N_t.

- Household either work H hours or not at all. Maybe due to fixed costs of going to work. Ideally like to work $N < H$ hours. This is not available.

- There exists a lottery which allows households to choose a probability of working, α_t. Chance then allocates some households to work and others to not.

- Let subscript 1 refer to those that are working and 2 to those who are not (unemployed).

  Expected utility prior to the draw is

  $$\alpha u(c_1, 1 - H) + (1 - \alpha) u(c_2, 1) \quad (1)$$

  recall endowment is 1 unit of time.

- Total consumption cannot exceed the per capita consumption level, c, across the economy so $\alpha c_1 + (1-\alpha)c_2 = c$ (2)
  Max (1) s.t. (2) $\Rightarrow$
  $$U_{c_1} = \lambda, U_{c_2} = \lambda \Rightarrow U_{c_1} = U_{c_2}$$
  where λ is lagrange multiplier ie. marginal utility of consumption must be equated across types
- Hansen assumes $U(c, L) = \ln c + A\ln(L)$
  $\Rightarrow c_1 = c_2$ [Amount of leisure consumed does not affect MU_c]
- Substituting in (1) $\Rightarrow$

$$\alpha_t u(c_t, 1-H) + (1-\alpha_t)u(c_t, 1)$$

$$\alpha_t \ln c_t + (1-\alpha_t)\ln c_t + \alpha_t A\ln(1-H) + (1-\alpha_t)A\ln(1)$$

$\Rightarrow \ln(c_t) + \alpha_t A\ln(1-H)$

- Also we know $H_t = 1 - l_t = \alpha_t H \Rightarrow \alpha_t = \frac{1-l_t}{H}$ (4)
  Substitute into prefs
  $\Rightarrow \ln(c_t) + \frac{1-l_t}{H} A\ln(1-H)$
  $\Rightarrow \ln(c_t) + B(1-l_t)$ (5) where $B = \frac{A}{H}\ln(1-H)$

- Practically, to use indivisible labour model, proceed as before just use (5) for preferences.
- Note for the rep. agent, preferences are linear in leisure $\Rightarrow \eta = 0$ and $\frac{1-N}{N\eta} = \infty$
- The micro elasticity is irrelevant
- It is optimal to allow unemployment
- The demand for labour from firms determines total hours worked (and employment).

  Hours worked $H_t = \alpha_t H + (1 - \alpha_t)0$
- Firms employ labour upto the point where the marginal product of hours worked equals the wage rate.
- But HH's are paid not for actual hours worked but for expected (average) hours worked (H_t)

  (Recall: sign ex-ante contracts)

  So budget constraint for HH is

  $c_t + I_t \leq W_t \alpha_t H + R_t K_t \quad (3)$
- Indivisible labour model with more general preferences.

$$U(c, L) = \frac{1}{1-\sigma}\left\{[cv(L)]^{1-\sigma} - 1\right\} \quad (6)$$

$$\sigma \neq 1$$

  As before the efficient allocation implies [max1s.t.2]

$$U_{c_1} = U_{c_2}$$

Here $U_{c_1} = [c_{1v}(1-H)]^{-\sigma} v(1-H) = U_{c_2}$

$$= [c_2 v(1)]^{-\sigma} v(1)$$

$\Rightarrow= [c_{1v}(1-H)]^{-\sigma} = [c_2 v(1)]^{-\sigma} \frac{v(1)}{v(1-H)}$

$\Rightarrow \frac{1}{c_1} = \frac{1}{c_2}[\frac{v(1)}{v(1-H)}]^{\frac{1}{\sigma}-1}$

OR $\quad c_1 = c_2[\frac{v(1-H)}{v(1)}]^{\frac{1-\sigma}{\sigma}} \quad (7)$

if $\sigma > 1 \Rightarrow c_{1>} c_2$ since v is an increasing function.

As more HH are allocated to work $\alpha_t \uparrow$,aggregate c will $\uparrow$ even though individual c stays the same.

- (7) can be used to figure out the appropriate preferences to be used by the representative agent.

  They are:

  $U(c, L)=\frac{1}{1-\sigma}\{c^{1-\sigma} v^*(L)^{1-\sigma} - 1\}$

  $v^*(L) = [(\frac{1-L}{H})v(1-H)^{\frac{1-\sigma}{\sigma}} + (1 - \frac{1-L}{H})v(1)^{\frac{1-\sigma}{\sigma}}]^{\frac{\sigma}{1-\sigma}} (8)$

  King and Rebelo show that even in this case, labour supply elasticity is infinite.

  Introduction of Government Spending

- Adding other shocks could help problem of $cor(\frac{Y}{n}, n)$ being too high.

  GRAPH

A positive tech shock shifts L^d up and vice-versa. $\Rightarrow$ both w and n move together.

- Any shocks that shift L^s could help reduce this correlation.

  If G does not affect utility or production $\Rightarrow$ pure negative income effect. $\Rightarrow$ L^s shifts.

  Now w and n move in opposite directions. (GRAPH)

  Government spending is governed by

  $$\log g_{t+1} = (1-\lambda)\log g^{-} + \lambda \log g_t + \mu_t$$

  $$\lambda \epsilon (0,1) \quad \mu_t \sim iid(o, \sigma_\mu) \quad (9)$$

  μ_t is independent of tech shock

  G is financed by lumpsum taxation

  $$max E\, \Sigma_{t=0}^{\infty} \beta^t u(c_t, l_t) (10)$$

  $$s.t\ l_t + h_t = 1 \qquad (11)$$

  $$y_t = \exp(z_t^{-}) k_t^{\theta} n_t^{1-\theta} \qquad (12)$$

  $$z_t^{-} = z_t + z^{-}t \qquad (13)$$

  $$z_{t+1} = \rho z_t + \varepsilon_t \quad let\ z^{-} = 0$$

  $$k_{t+1} = (1-\delta)k_t + I_t \quad (14)$$

  $$g_t + c_t + I_t = Y_t \qquad (15)$$

  Combining (12), (14) and (15).

  $$g_t + c_t + k_{t+1} - (1-\delta)k_t = z_t k^{\theta} n_t^{1-\theta} (16)$$

Plug (11) into (10) and maximize st (16)

max $\Sigma_{t=0}^{\infty}(\log c_t + A\log(1-N_t)$

$+\Sigma_{t=0}^{\infty}\beta^t\lambda_t[g_t + c_t + k_{t+1} - (1-\delta)k_t - z_t k^0 N_t^{1-\theta}]$

$\Rightarrow F.O.C \qquad \frac{1}{c_t} = \lambda_t \qquad (18)$

(19) $\qquad \frac{+A}{1-n_t} = \lambda_t z_t k_t^{\theta}(1-\theta)N_t^{-\theta}$

(20) $\qquad +\beta^t\lambda_t = +\beta^{t+1}\lambda_{t+1}[1-\delta+z_{t+1}N_{t+1}^{1-\theta}\theta k_{t+1}^{\theta-1}]$

So F.O.C are the same but constraint is different.

$\uparrow$ in G_t is pure drain on Y

$\Rightarrow$ neg. wealth effect [leisure is normal good]

$\Rightarrow N_t\uparrow\Rightarrow$ Negative relationship between N and $\frac{Y}{N}$.

$\Rightarrow$ Size of wealth effect depends on persistence of G shock.

OLS regression based on 16 yields

$\lambda = .96$ and $\sigma_\mu = .021$

Average $\frac{G_t}{Y_t} = .22 = g^-$

Results: Model with G and z shocks

$\sigma_y = 1.24$

$\sigma_h/\sigma_y = .55 \qquad (.78-.96)$

$\sigma_h/\sigma_w = .90 \qquad (1.37-2.15)$

$Cor(h,w) = .49 \quad (.07-(-.14))$

$Cor(h,w)$ significantly better than .94 in baseline model.

- Also see Christiano-Eichenbaur (92) for more details.